ON SOME CONDITIONAL EXTREMAL PROBLEMS IN THE CLASS OF ENTIRE FUNCTIONS OF FINITE DEGREE

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.512.6

MATHEMATICS

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ON SOME CONDITIONAL EXTREMAL PROBLEMS IN THE CLASS OF ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician S. N. Bernstein on 7 V 1965)

Denote by $\overset{+}{W}{}_\sigma^{(p)}$ $(p \geq 1)$ the class of real entire functions $\varphi_\sigma(z)$ of degree σ, nonnegative on the real axis, with norm

$$\|\varphi_\sigma\|_{L_p} = \left(\int_{-\infty}^{\infty} [\varphi_\sigma(x)]^p\, dx\right)^{1/p} < \infty. \tag{1}$$

Let $F_j(\varphi_\sigma)$ $(j = 1, \dots, \nu)$ be linear functionals defined on the set $\overset{+}{W}{}_\sigma^{(p)}$.

We shall consider functions $\varphi_\sigma(x)$ subject to the conditions

$$F_j(\varphi_\sigma) = A_j \qquad (j = 1, \dots, \nu), \tag{2}$$

where A_j are given real numbers, with not all A_j equal to zero; the conditions (2) are assumed admissible, i.e. compatible and not contradicting the membership of the functions $\varphi_\sigma(x)$ in the class $\overset{+}{W}{}_\sigma^{(p)}$ $(p \geq 1)$.

In the present article upper and lower estimates are indicated for the quantity

$$\|\varphi_\sigma^*\|_{L_p} = \inf_{\varphi_\sigma \in \overset{+}{W}{}_\sigma^{(p)}} \|\varphi_\sigma\|_{L_p}. \tag{3}$$

The function $\varphi_\sigma^*(x) \in \overset{+}{W}{}_\sigma^{(p)}$ will be called the extremal function of our problem in the class $\overset{+}{W}{}_\sigma^{(p)}$. The founder of this type of problem in the metric $C(-\infty, \infty)$ is S. N. Bernstein ([1]).

It is obvious that $\varphi_\sigma^* \neq$ const, since otherwise one of the conditions (1) or (2) would not be satisfied.

It is known ([1,2]) that if $\varphi_\sigma(x) \in \overset{+}{W}{}_\sigma^{(p)}$ and $p_1 > p \geq 1$, then an inequality of S. M. Nikolskii type holds:

$$\|\varphi_\sigma\|_{L_{p_1}} \leq \left(\frac{s\sigma}{2\pi}\right)^{1/p-1/p_1} \|\varphi_\sigma\|_{L_p}, \tag{4}$$

where $s = |[-p/2]|$ is the least integer not less than $p/2$. Therefore, denoting by $\Phi_\sigma(x)$ a function belonging to $\overset{+}{W}{}_\sigma^{(1)}$, and taking into account that $\overset{+}{W}{}_\sigma^{(1)} \subset \overset{+}{W}{}_\sigma^{(p)} \subset \overset{+}{W}{}_\sigma^{(\infty)}$, we have

$$\|\varphi_\sigma^*\|_{L_p} \leq \|\Phi_\sigma^*\|_{L_p} \leq \left(\|\Phi_\sigma^*\|_C\right)^{(p-1)/p} \left(\|\Phi_\sigma^*\|_L\right)^{1/p}, \tag{5}$$

where a function $f_\sigma(z) \in \overset{+}{W}{}_\sigma^{(\infty)}$ is considered bounded on $(-\infty, \infty)$.

Consequently, the following is valid.

Theorem. If $f_\sigma^*(x)$ and $\Phi_\sigma^*(x)$ are the extremal functions of our problem, respectively, in the classes $\overset{+}{W}{}_\sigma^{(\infty)}$ and $\overset{+}{W}{}_\sigma^{(1)}$, then in the class $\overset{+}{W}{}_\sigma^{(p)}$ $(1 < p < \infty)$ there exists a function $\varphi_\sigma^*(x)$, subject to the conditions (2),

with the least norm $\|\varphi_\sigma^*\|_{L_p}$, for which the inequalities

$$\left(\frac{2\pi}{s\sigma}\right)^{1/p} |f_\sigma^*|_C \leq \|\varphi_\sigma^*\|_{L_p} \leq \|\Phi_\sigma^*\|_{L_p} \leq \left(\|\Phi_\sigma^*\|_C\right)^{(p-1)/p} \left(\|\Phi_\sigma^*\|_L\right)^{1/p}. \tag{6}$$

As is known ([4]), if $\Phi_\sigma(x) \in \overset{+}{W}{}_\sigma^{(1)}$, then for $\nu \leq 2$

$$\Phi_\sigma^*(x) = [\Psi_{\sigma/2}(x)]^2, \tag{7}$$

where $\Psi_{\sigma/2}(x)$ is a certain real entire function of degree $\sigma/2$ from the class $W_\sigma^{(2)}$ (see ([2]), p. 38). Then, by the well-known Wiener–Paley theorem, we have

$$\Psi_{\sigma/2}(x) = \int_{-\sigma/2}^{\sigma/2} \gamma(t) e^{ixt}\, dt, \tag{8}$$

where $\gamma(t) \in L_2[-\sigma/2, \sigma/2]$, and

$$\int_{-\infty}^{\infty} [\Psi_{\sigma/2}(x)]^2\, dx = 2\pi \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2\, dt. \tag{9}$$

Writing the formal expansion of the function $\gamma(t)$ in the Legendre polynomials $\{\hat{P}_n(t)\}_0^\infty$ normalized on $[-\sigma/2, \sigma/2]$,

$$\gamma(t) \sim \sum_{k=0}^{\infty} b_k \hat{P}_k(t), \tag{10}$$

where

$$b_k = \int_{-\sigma/2}^{\sigma/2} \gamma(t) \hat{P}_k(t)\, dt \qquad (k = 0, 1, 2, ...), \tag{11}$$

we obtain

$$\|\Phi_\sigma^*\|_L = \int_{-\infty}^{\infty} [\Psi_{\sigma/2}(x)]^2\, dx = 2\pi \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2\, dt = 2\pi \sum_{k=0}^{\infty} b_k^2. \tag{12}$$

Moreover,

$$\|\Phi_\sigma^*\|_C = \left[\sup_{-\infty<x<\infty} |\Psi_{\sigma/2}(x)|\right]^2 \leq \left(\int_{-\sigma/2}^{\sigma/2} |\gamma(t)|\, dt\right)^2 \leq \sigma \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2\, dt = \sigma \sum_{k=0}^{\infty} b_k^2. \tag{13}$$

Finally, denote by $\widetilde{B}_\sigma$ the class of entire functions $g_\sigma(z)$ belonging to the class B_σ (B_σ is the class of entire functions of degree σ, bounded on the entire real axis ([1]), and satisfying the conditions (2)). Let $g_\sigma^*(x)$ be an entire function from the class $\widetilde{B}_\sigma$ with the least norm,

$$\|g_\sigma^*\|_C = \inf_{g_\sigma \in \widetilde{B}_\sigma} \|g_\sigma\|_C.$$

It is not difficult to observe that $\|f_\sigma^*\|_C \geq |f_\sigma^*(0)|$ and, moreover,

$$|g_\sigma^*(0)| \leq \|g_\sigma^*\|_C \leq \|f_\sigma^*\|_C. \tag{14}$$

Thus, inequality (6) (taking into account (12), (13), and (14), for $\nu \leq 2$) takes the form

$$\left(\frac{2\pi}{s\sigma}\right)^{1/p} \|g_\sigma^*\|_C \leq \|\varphi_\sigma^*\|_{L_p} \leq \sigma \left(\frac{2\pi}{\sigma}\right)^{1/p} \sum_{k=0}^{\infty} b_k^2. \tag{15}$$

Let us now consider some particular cases of inequalities (6) or (15).

1. Let $\nu = 1$ and let relation (2) be

$$\varphi_\sigma(0) = 1. \tag{16}$$

S. N. Bernstein [1] proved that in this case $g^*_\sigma(x) = \sin\sigma x/\sigma x$ is an extremal function in the class $\widetilde{B}_\sigma$ and $\|g^*\|_C = 1$. On the other hand, as shown in [2], in this case

$$b_0 = \frac{1}{\sqrt{\sigma}}, \qquad b_k = 0 \quad (k = 1, 2, \ldots); \qquad \Phi^*_\sigma(x) = \left(\frac{\sin\sigma x/2}{\sigma x/2}\right)^2. \tag{17}$$

Therefore inequality (15) takes the form

$$(2\pi/s\sigma)^{1/p} \le \|\varphi^*_\sigma\|_{L_p} \le (2\pi/\sigma)^{1/p}. \tag{18}$$

2. Let now $\nu = 2$, and let the relations (2) have the form

$$\varphi_\sigma(0) = 1, \qquad \varphi'_\sigma(0) = 0. \tag{19}$$

Then for $\Phi^*_\sigma(x) \in \overset{+}{W}{}^{(1)}_\sigma$ (under conditions (19)) the coefficients b_k are determined by the same formulas (17).

On the other hand, it is known [3] that in the class $\widetilde{B}_\sigma$, under conditions (19), the extremal function is $\cos\sigma x$, with norm equal to 1. Therefore, as in the preceding case, we have

$$(2\pi/s\sigma)^{1/p} \le \|\varphi^*_\sigma\|_{L_p} \le (2\pi/\sigma)^{1/p}. \tag{18'}$$

It is interesting to note that for $p = 2$ (then $s = 1$) inequalities (18) and (18′) become the equality

$$\|\varphi^*_\sigma\|_{L_2} = (2\pi/\sigma)^{1/2};$$

moreover, for all $p \in [1, 2]$ the equality

$$\|\varphi^*_\sigma\|_{L_p} = (2\pi/\sigma)^{1/p}$$

holds.

Received
20 IV 1965

[1] S. N. Bernstein, *Collected Works*, 1, 1952, p. 22.
[2] B. A. Rymarenko, DAN, 161, No. 4 (1965).
[3] N. I. Akhiezer, *Matematicheskii sbornik*, 31 (73), No. 2 (1952).

CITED LITERATURE

Note: Figure translations are in progress. See original paper for figures.