ON THE MUTUAL GROWTH OF THE COEFFICIENTS OF ONE CLASS OF p-VALENT FUNCTIONS

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.54

MATHEMATICS

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ON THE MUTUAL GROWTH OF THE COEFFICIENTS OF ONE CLASS OF p-VALENT FUNCTIONS

(Presented by Academician V. I. Smirnov, 19 XI 1965)

For the class $S^{\square}$ of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular and univalent in the mean with respect to area in the disk $|z| < 1$, Hayman ([1]) proved that

$$\big||a_{n+1}| - |a_n|\big| < A, \qquad n \geq 2, \tag{1}$$

where A is an absolute constant. The order of estimate (1) is sharp.

For the class S^* of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, univalent and starlike in the disk $|z| < 1$, estimate (1) (with $A < 100$) was obtained by G. M. Goluzin ([2]) in 1946.

Hayman ([3]) posed the question whether

$$\big||a_{n+1}| - |a_n|\big| \to 0 \quad \text{as } n \to \infty$$

for every function of the class $S^{\square}$, other than functions of a certain special form.

In this direction, for the class S^* the following result of Pommerenke ([4]) is known: every function $f(z) \in S^*$ either has the form

$$\frac{z}{(1 - e^{-i\theta_1} z)(1 - e^{-i\theta_2} z)}, \qquad \theta_1, \theta_2$$

real, or there exists $\delta = \delta(f) > 0$ such that

$$|a_{n+1}| - |a_n| = O(n^{-\delta}).$$

Lucas ([3]) generalized estimate (1) to the class of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, regular and p-valent in the mean with respect to area ($p \geq 1$) in the disk $|z| < 1$, proving that

$$\big||a_{n+1}| - |a_n|\big| < A(p)\mu_p n^{2p-2}, \qquad n \geq 1, \tag{2}$$

where $\mu_p = \max_{0\leq\nu\leq p} |a_\nu|$, $A(p)$ is a constant depending only on p. The order of estimate (2) is sharp.

In the present paper, for a certain class of p-valent functions, a result analogous to the above-mentioned result of Pommerenke is obtained.

Let $F(p)$ (p a fixed natural number) be the class of functions $g(z)$ representable in the disk $|z| < 1$ by the formula

$$g(z) = [\varphi(z)]^p z^{q-p} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right)(1 - z\overline{\alpha}_s),$$

where q is an integer, $1 \leq q \leq p$, $\varphi(z) \in S^*$, $0 < |\alpha_s| < 1$, $s = 1, 2, \dots, p-q$; $F(1) \equiv S^*$.

Let $F(p,q)$ be the subclass of all functions from $F(p)$ of the form

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$$

with fixed q.

Bender [5] showed that all functions of the class $F(p,q)$ are p-valent. For $p > q \geqslant 1$, the class $F(p,q)$ contains as a subclass the class $S(p,q)$ [5] of functions of the form

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n,$$

regular in the disk $|z| < 1$ and satisfying the condition: for each function $g(z)$ there exists ρ, $0 < \rho < 1$, such that

$$\mathrm{Re}\left[\frac{zg'(z)}{g(z)}\right] > 0, \qquad \rho < |z| < 1,$$

$$\int_0^{2\pi} \mathrm{Re}\left[\frac{zg'(z)}{g(z)}\right] d\theta = 2\pi p, \qquad z = re^{i\theta}, \quad \rho < r < 1.$$

For $p = q$ we have $F(p,p) \equiv S(p,p)$.

Theorem. If

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in F(p,q)$$

and $p > 1$, and if

$$g(z) \not\equiv \frac{z^q}{(1-e^{-i\theta}z)^{2p}} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right)(1 - z\overline{\alpha}_s),$$

then there exists $\delta = \delta(g) > 0$ such that

$$|a_{n+1}| - |a_n| = O\left(n^{2p-2-\delta}\right), \qquad n \geqslant q.$$

It is not difficult to show that there exist functions of the class $F(p,q)$, of the form excluded in the theorem, for which

$$|a_{n+1}| - |a_n| \sim K n^{2p-2} \quad \text{as } n \to \infty,$$

where K is a constant independent of n.

In the proof of the theorem the following result is established:

Lemma. Let $m \geqslant 1$, m an integer, and let

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in F(p,q).$$

If

$$g(z) \not\equiv z^q \prod_{k=1}^{m+1} (1 - e^{-i\theta_k} z)^{-2p/(m+1)} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right)(1 - z\overline{\alpha}_s)$$

(θ_k real), then there exist complex numbers c_k, $k = 0, 1, \dots, m$, $|c_0| = |c_m| = 1$, and $\delta > 0$, depending only on m and g, such that for $n \geqslant q$ we have

$$\left|c_0 n^2 a_n + c_1 (n+1)^2 a_{n+1} + \dots + c_m (n+m)^2 a_{n+m}\right| = O\left(n^{2/(m+1)+2p-1-\delta}\right).$$

For $p = 1$, an analogous result was obtained by Pommerenke [4] for the class of close-to-convex functions.

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Received
11 XI 1965

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Note: Figure translations are in progress. See original paper for figures.