DISTRIBUTION OF EIGENVALUES FOR SINGULAR DIFFERENTIAL OPERATORS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.948.35

MATHEMATICS

A. G. KOSTYUCHENKO

DISTRIBUTION OF EIGENVALUES FOR SINGULAR DIFFERENTIAL OPERATORS

(Presented by Academician I. G. Petrovskii, January 17, 1966)

I. Consider a strongly elliptic operator L in the space of N-dimensional vector-functions $u(x) = \{u_1(x), \dots, u_N(x)\}$

$$L = (-1)^m \sum_{j_1+\dots+j_n=2m} A^{j_1\cdots j_n}(x) \frac{\partial^{2m}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} + L_1\left(x, \frac{\partial}{\partial x}\right) + Q(x) =$$

$$= L_0 + L_1 + Q(x),$$

where $x = (x_1, \dots, x_n)$, $-\infty < x_j < \infty$, $j = 1, \dots, n$, $A^{j_1\cdots j_n}(x)$ are symmetric matrices. By $L_1(x, \partial/\partial x)$ is denoted a linear differential operator of order $< 2m$, generally speaking with complex-valued coefficients; $Q(x)$ is the operator of multiplication by the symmetric matrix $Q(x)$.

Number the characteristic roots $\lambda_i(x)$ of the matrix $Q(x)$ in increasing order:

$$\lambda_1(x) \le \lambda_2(x) \le \dots \le \lambda_n(x).$$

We shall assume that the following conditions are satisfied:

a) the matrix $A(s, x)$, defined by the equality

$$A(s, x) = (-1)^{m+1} \sum_{j_1+\dots+j_n=2m} A^{j_1\cdots j_n}(x)(is_1)^{j_1} \cdots (is_n)^{j_n},$$

is Hermitian-symmetric for all x and real $s = (s_1, \dots, s_n)$; its characteristic roots $\sigma_j(s, x)$ satisfy the inequality

$$\sigma_j(s, x) \le -\delta |s|^{2m}, \qquad \delta > 0;$$

b) $\|A(\xi, x) - A(\xi, y)\| \le c|x - y|^\gamma |\xi|^{2m}$, $0 < \gamma < 1$, for $|x - y| \le 1$.

The characteristic roots $\lambda_i(x)$ of the matrix $Q(x)$ are subject to the conditions:

1) $\|Q(x) - Q(y)\| \leq B\lambda_1^\alpha(y)|x-y|$, if $|x-y| \leq 1$, $\alpha < 1+$

$$+1/2m;$$

2) $\lambda_n(x) \leq B_1 \exp(c^*|x-y|\lambda^{1/2m}(y))$, if $|x-y| > 1$; here c^* is some constant depending on the matrix $A(\xi, x)$;
3) $\lambda_1(x) \leq B\lambda_1(y)$, if $|x-y| \leq 1$;
4) $\lambda_n(x) \leq \lambda_1^k(x)$ for some $k \leq 1$;
5) the elements $p_{ik}^{j_1 \cdots j_n}(x)$ of the matrix $P^{j_1 \cdots j_k}(x)$ $(j_1 + \cdots + j_n = |j|)$ of the operator $L_1(x, \partial/\partial x)$, standing at $\partial^{|j|}/\partial x_1^{j_1} \cdots \partial x_n^{j_n}$, satisfy the condition

$$\left|p_{ik}^{j_1 \cdots j_n}(x)\right| \leq \lambda_1^{(2m-|j|)/2m-\varepsilon_0}(x), \qquad \varepsilon_0 > 0.$$

We shall be interested in asymptotic formulas for the function $N(\lambda)$—the number of eigenvalues of the operator L not exceeding λ. With this

for this purpose one can study the Green function (more precisely, the Green matrix-function) $G(x, y, t)$ of the Cauchy problem for the corresponding parabolic system:

$$\partial u/\partial t = -Lu = -(L_0 + L_1 + Q(x))u. \tag{1}$$

The function $N(\lambda)$ will be connected, as is easily verified, with $G(x, y, t)$ in the following way:

$$\operatorname{tr} \int_{-\infty}^{\infty} G(x, x, t)\, dx = \int_0^\infty e^{-\lambda t}\, dN(\lambda).$$

Thus it is necessary to study $G(x, y, t)$ for small t, uniformly in R_n. For a single elliptic equation this question was studied by the author in ([1]).

Let $G_0(x-y, \eta, t)$ denote the Green function of the following system with "frozen" coefficients:

$$\frac{\partial U}{\partial t} = (-1)^{m+1} \sum_{|j|=2m} A^{j_1 \cdots j_n}(\eta) \frac{\partial^{2m} U}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} - Q(\eta) U.$$

Theorem 1. *If conditions a), b) and 1)–5) are satisfied, then for the Green function $G(x, y, t)$ of system (1) one has the representation*

$$G(x, y, t) = G_0(x-y, y, t) + S(x, y, t), \tag{2}$$

where the matrix $S(x,y,t)$ satisfies the inequality

$$\|S(x,y,t)\| \leq \frac{c_1}{t^{(n-\varepsilon)/2m}} \exp\left(-ct\lambda_1(y) - c\frac{|x-y|^{2m'}}{t^{1/(2m-1)}}\right) + \frac{c_1}{\lambda_1^l(y)} \exp\left(-c\frac{|x-y|^{2m'}}{t^{1/(2m-1)}}\right).$$

Here $1/2m + 1/2m' = 1$; the number $l > 0$ can be taken arbitrarily large; c, c_1 are constants.

The proof of the theorem is carried out according to the scheme of the author' s work ([1]).

We note that, although representation (2) contains rather rich information, for all the systems under consideration one cannot obtain more or less explicit formulas for the principal term of $N(\lambda)$. This is connected with the fact that the matrices $A(s,x)$ and $Q(x)$ do not commute with one another. We shall now consider two special cases.

Theorem 2. *Let the self-adjoint operator L have the form*

$$Ly = (-1)^m y^{(2m)} + L_1\left(x_1, \frac{d}{dx}\right) y + Q(x)y,$$

where $y(x) = \{y_1(x), \dots, y_n(x)\}$, $x \in R_1$, the coefficients of the operator L_1 and the matrix $Q(x)$ satisfy conditions 1)–5). In addition, $\lambda^{-r}(x) \in L_1(-\infty,\infty)$ for some $r > 0$, and the function $M(\lambda)$, defined by the equality

$$M(\lambda) = \frac{1}{\pi}\sum_{i=1}^{n}\int_{\lambda_i(x)<\lambda} (\lambda - \lambda_i(x))^{1/2m}\, dx$$

*satisfies some Tauberian condition *. Then the formula holds:*

$$N(\lambda) \sim \frac{1}{\pi}\sum_{i=1}^{n}\int_{\lambda_i(x)<\lambda} (\lambda - \lambda_i(x))^{1/2m}\, dx.$$

* For example, the Tauberian condition in the theorem of B. M. Korenblum ([2]) $\lambda M'(\lambda) < a_0 M(\lambda)$.

The proof of this theorem follows from Theorem 1, since in this case

$$\operatorname{tr}\int_{-\infty}^{\infty} G_0(0,x,t)\, dx = \frac{1}{2\pi}\frac{1}{t^{1/2m}}\left\{\int_{-\infty}^{\infty} e^{-s^{2m}}\, ds\right\} \operatorname{tr}\int_{-\infty}^{\infty} e^{-Q(x)t}\, dx = \int_{-\infty}^{\infty} e^{-\lambda t}\, dM(\lambda).$$

Theorem 3. Let the self-adjoint operator L have the form

$$Ly = (-1)^m A \frac{d^{2m}y}{dx^{2m}} + L_1\left(x, \frac{d}{dx}\right) y + x^{sRy},$$

where $x \in R_1$; A, R are positive definite constant matrices. The coefficients $p^j(x)$ of the operator $L_1(x, d/dx)$, which has order $< 2m$, standing at the j-th derivative, satisfy the inequality

$$\|p^j(x)\| \leq c_j |x|^{s(2m-|j|)/2m-\varepsilon_0}, \qquad \varepsilon_0 > 0,$$

for large x. Then the formula holds:

$$N(\lambda) = \frac{S}{2\pi\Gamma(1/2m + 1/s + 1)} \lambda^{1/2m+1/s}, \qquad S = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} \operatorname{tr} \exp\{-\xi^{2m} A - \sigma^s R\}\, d\xi.$$

Condition 1°, which the matrix $Q(x)$ must satisfy, is very restrictive, since it essentially shows that the largest root can grow only in a definite way, in accordance with the growth of $\lambda_1(x)$. Roughly speaking, if $\lambda_n(x) \leq c\lambda_1^k(x)$, then $k < 1 + 1/2m$.

Theorem 4. Let the matrix $Q(x)$ of the operator L (self-adjoint)

$$L = (-1)^m \frac{d^{2m}}{dx^{2m}} + L_1\left(x, \frac{d}{dx}\right) + Q(x)$$

be such that the elements of the unitary matrix $T(x)$ reducing Q to diagonal form have $2m$ bounded derivatives for all $x \in R_1$. Then the formula holds

$$N(\lambda) \sim \frac{1}{\pi} \sum_{i=1}^{n} \int_{\lambda_i(x)<\lambda} \{\lambda - \lambda_i(x)\}^{1/2m}\, dx,$$

if the following conditions are fulfilled:

1) $|\lambda_i(x) - \lambda_i(y)| \leq B\lambda_i^\alpha(y)|x-y|$, $\lambda_i(x) \leq c\lambda_i(y)$ for $|x-y| \leq 1$, $\alpha < 1 + 1/2m$;
2) $\lambda_i(x) \leq B \exp(c^* \lambda_i^{1/2m}(y)|x-y|)$ for $|x-y| > 1$ and some c^*;
3) $\lambda_i^{-l}(x) \in L_1(-\infty, \infty)$ for some $l > 0$;
4) the Tauberian condition is satisfied for the function $M(\lambda)$;
5) for the coefficients $p^j(x)$ of the operator $L_1(x, d/dx)$ the subordination condition is satisfied:

$$\|p^j(x)\| \leq c\lambda_1(x)^{(2m-|j|)/2m-\varepsilon_0}, \qquad \varepsilon_0 > 0.$$

II. In [1] and above, the case was considered where only the "free" coefficient affected the asymptotics of $N(\lambda)$. The coefficients standing at the other lower derivatives played a subordinate role. We now formulate one result pertaining to another situation.

Consider the self-adjoint ordinary operator L

$$Ly = (-1)^m y^{(2m)} + p_{2m-2}(x) y^{(2m-2)} + \dots + p_0(x) y + L_1\left(x, \frac{d}{dx}\right) y,$$

where $y(x)$ is a scalar function, $x \in R_1$.

Assume that the characteristic polynomial

$$P(s, x, \lambda) = s^{2m} + p_{2m-2}(x)(is)^{2m-2} + \dots + p_0(x) + \lambda$$

for all sufficiently large x and $\lambda > 0$ and all s are positive. Further suppose that the coefficients $p_i(x)$ of the operator L and the roots $s_i(x, \lambda)$ of the characteristic polynomial $P(s, x, \lambda)$ are such that the following conditions are satisfied:

$$1^\circ. \quad B \leqslant \left|\frac{s_k(x,\lambda)}{s_j(x,\lambda)}\right| \leqslant A, \quad B \leqslant \left|\frac{\operatorname{Im} s_k(x,\lambda)}{s_j(x,\lambda)}\right| \leqslant A.$$

$$2^\circ. \quad |p_j(x) - p_j(\xi)| \leqslant c|x-\xi| p_j^{\alpha_j}(x), \quad p_j(x) \leqslant c p_j(\xi) \quad \text{for } |x-\xi| \leqslant 1, \quad \alpha_j < 1 + \frac{1}{2m-j}.$$

$$3^\circ. \quad |p_j(x)| \leqslant c \exp\left(c^* |x-\xi| p_0^{1/2m}(\xi)\right) \quad \text{for } |x-\xi| \geqslant 1 \text{ and some } c^*.$$

$$4^\circ. \quad q_0^{-(2m-1)/2m}(x) \in L_1(-\infty, \infty).$$

$$5^\circ. \quad |P'(s_k(x,\lambda))|^{-1} \leqslant c(\lambda + p_0(x))^{-(2m-1)/2m}.$$

We now introduce the function

$$\rho(\lambda) = i \sum_{k=1}^{m} \int_{-\infty}^{\infty} \frac{dx}{P'(s_k(x,\lambda))}.$$

As is seen, $\rho(\lambda)$ is an analytic function in the plane $\lambda = \xi + i\tau$ with a slit along the positive semiaxis. Let $\sigma(t)$ be defined by the equality

$$\sigma(t) = \frac{1}{\pi} \lim_{\tau \to 0} \int_0^t \operatorname{Im}\{\rho(-\xi + i\tau)\}\, d\xi, \qquad \xi > 0, \ \tau > 0.$$

Theorem 5. *Let conditions* 1°–5° *be satisfied, let the function* $\sigma(t)$ *satisfy the Tauberian condition, and let the coefficients* $q_i(x)$ *of the operator* $L_1(x, d/dx)$ *be subordinate to* $p_0(x)$*, i.e.*

$$|q_i(x)| \leqslant c p_0^{(2m-i)/2m-\varepsilon_0}(x), \qquad \varepsilon_0 > 0.$$

Then the asymptotic formula holds

$$N(\lambda) \sim \sigma(\lambda).$$

To prove the theorem we find the asymptotics of the kernel $K(x, \xi, \lambda)$ of the resolvent of the operator L for large λ. Then we use a Tauberian theorem.

For the operator of the 4th order

$$Ly = y^{\mathrm{IV}} - 2(p_2(x)y')' + p_0(x)y + L_1\left(x, \frac{d}{dx}\right) y,$$

as is not hard to verify, all conditions $1^\circ, 2^\circ, 7^\circ$ will be satisfied if, for sufficiently large $|p_2'(x)| \leqslant c(p_0(x) + \lambda_0)$ for some λ_0 and $c < 1$. In this case

$$N(\lambda) \sim \frac{1}{\pi} \int_{p_0(x)<\lambda} \sqrt{-p_2(x) + \sqrt{p_2^2(x) + (\lambda - p_0(x))}}\, dx.$$

Moscow State University
named after M. V. Lomonosov

Received
11 I 1966

CITED LITERATURE

[1] A. G. Kostyuchenko, DAN, **158**, No. 1, 41 (1964).
[2] B. I. Korenblyum, DAN, **88**, No. 5, 745 (1958).

Note: Figure translations are in progress. See original paper for figures.