REPRESENTATIONS OF TOPOLOGICAL SEMIGROUPS BY CONTINUOUS TRANSFORMATIONS

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REPRESENTATIONS OF TOPOLOGICAL SEMIGROUPS BY CONTINUOUS TRANSFORMATIONS

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1. A **representation** of a topological semigroup A_t (2) in a topological semigroup $A'_{t'}$ will mean a continuous homomorphic mapping of A_t into $A'_{t'}$. We shall say that a topological semigroup A_t is **representable** in a class of topological semigroups Σ if there exists a representation of A_t in some semigroup from the class Σ. If there exists a topological isomorphism (2) of A_t onto some topological semigroup from the class Σ, then we shall say that A_t is i-representable in the class Σ.

In this paper it is established that every topological semigroup is i-representable in the class of topological semigroups of continuous transformations of topological spaces (4). Sufficient conditions are found for an i-representation of topological semigroups in the class of semigroups of continuous transformations of bicompacts (7), and it is shown that complete regularity is a necessary condition for such a representation (5). Further, it is established that every bicompact topological semigroup is representable in the class of semigroups of continuous transformations of compact spaces (8), and at the same time in the class of semigroups of continuous transformations of compact subsets of the Hilbert cube (10). The sufficiency of the system of such representations (9) is shown, which makes it possible to solve Ulam' s problem (5) on the construction of a universal bicompact topological semigroup (11).

2. A semigroup A, on the set of elements of which a topology t is defined, will be denoted by A_t. In the case when the semigroup operation is continuous with respect to this topology, the semigroup A_t is called **topological**. The semigroup A_t is called **bicompact**, **completely regular**, etc., when its topological space is bicompact, completely regular, etc. The semigroups A_t and $A'_{t'}$ are called **topologically isomorphic** if there exists a mapping φ which is an isomorphism of the algebraic semigroups A and A' and a homeomorphism of their topological spaces.

Let A_t be a semigroup with topology t. An isomorphism φ of the (algebraic) semigroup A onto the semigroup A' induces on the set of elements of the semigroup A' a topology, which will be denoted by $\varphi(t)$.

3. Let X be a topological space with topology t, and let S be some semigroup of continuous transformations of X. The topology t induces on the semigroup S the bicompact-open topology (2). Namely, let $B_1, \dots, B_n; U_1, \dots, U_n$, where n is a finite number, be subsets of X; all B_i bicompact, and the U_i open. By $(B_1, \dots, B_n; U_1, \dots, U_n)$ we shall denote the set of all transformations $s \in S$ for which $\forall_{1 \le i \le n} sB_i \subset U_i$. The system of all sets $(B_1, \dots, B_n; U_1, \dots, U_n)$ may be taken as a base of open sets of the topology on S. This topology will everywhere below be denoted by t. In the case when X is a compact metric space, this topology on S coincides with the natural one generated by the metric on X.

If the semigroup $S^{\hat{t}}$ is topological, then we shall call it a **topological semigroup of continuous transformations** of the space X.

Note that the semigroup $S^{\hat{t}}$ is not always topological. In this connection, the following theorem is interesting and important.

Theorem 1. *If X is bicompact, then the semigroup $S^{\hat{t}}$ is topological.*

This theorem makes it possible to speak of the semigroup of continuous transformations of a bicompactum without indicating each time that it is topological.

4. Let A_t be a topological semigroup. Without restricting the generality of the arguments, henceforth we shall everywhere assume that A_t contains an identity. Otherwise the identity e could be adjoined to the semigroup A externally ([3]), and to the topological space A_t as an isolated point.

To each element $a \in A$ we assign the continuous transformation f_a of the space of the topological semigroup A_t:

$$\forall_{x \in A_t} f_a x = ax.$$

It is clear that, with respect to the operation of superposition of transformations, the set $F(A_t) = \bigcup_{a \in A} f_a$ is a semigroup of continuous transformations of the space of the topological semigroup A_t.

It is known that the mapping $\varphi(a) = f_a$ of the semigroup A onto the semigroup $F(A_t)$ is an isomorphism ([3]). This isomorphism induces on the semigroup $F(A_t)$ the topology $\varphi(t)$ (§ 2). On the other hand, on the semigroup $F(A_t)$ one can construct the bicompact-open topology $\hat{t}$ (§ 3).

Lemma 1. *On the semigroup $F(A_t)$ the topologies $\varphi(\hat{t})$ and $\hat{t}$ coincide.*

From Lemma 1 we immediately obtain

Corollary. *The semigroup $F(A_t)$ is topological.*

From the same lemma it follows that

Theorem 2. *Every topological semigroup is i-representable in the class of topological semigroups of continuous transformations of topological spaces.*

Corollary. *A bicompact topological semigroup is i-representable in the class of semigroups of continuous transformations of bicompacta.*

5. The question naturally arises of the i-representation of topological semigroups in the class of semigroups of continuous transformations of bicompacta. First of all, let us note that not every topological semigroup is i-representable in this class, as is shown by the following

Theorem 3. *A semigroup of continuous transformations of a bicompactum is completely regular.*

6. We shall now find some sufficient conditions for the possibility of an i-representation of topological semigroups in the class of semigroups of continuous transformations of bicompacta. We shall need a number of auxiliary assertions.

Lemma 2. *Let X^* be the maximal bicompact extension of the topological space X. If B is a closed set and U an open set in X^*, and $B \cap X$ and $X \setminus (U \cap X)$ are functionally separable, then $B \subset U$.*

From the results ([1,4]) one can obtain

Lemma 3. *Let X be a completely regular topological space, X^* its maximal bicompact extension, and f a continuous mapping of X into X^*. Then f can be extended to a continuous mapping f^* of the bicompactum X^* into itself.*

7. Let U be open, Y an arbitrary subset of the space of the topological semigroup A_t, and $a \in A$. We shall call the topological semigroup A_t **strongly continuous on the left** if whenever aY and $A_t \setminus U$ are functionally separable, there exists a neighborhood V

point a, such that for any $a' \in V$ the closed sets $\overline{a'Y}$ and $A_t \setminus U$ are functionally separated.

Lemma 4. If A_t is a completely regular strongly left-continuous topological semigroup, U is open, Y is an arbitrary subset of its topological space, and $f_a \in F(A_t)$ (Sec. 4) is such that $\overline{f_{aY}}$ and $A_t \setminus U$ are functionally separated, then there exists an open neighborhood V of the element f_a in $\hat{F}_t(A_t)$ such that, for any $f \in V$, the closed sets $\overline{fY}$ and $A_t \setminus U$ are functionally separated.

With the aid of Lemmas 2, 3, and 4 one proves

Theorem 4. A completely regular strongly left-continuous topological semigroup is i-representable in the class of semigroups of continuous transformations of bicompacta.

Sec. 8. We now turn to the question of representations of bicompact topological semigroups.

In what follows we shall need the following

Lemma 5. If a semigroup $A'_{t'}$, with separable topology t', is the image of a bicompact topological semigroup A_t under a continuous homomorphic mapping, then $A'_{t'}$ is a bicompact topological semigroup.

Let A_t be a bicompact topological semigroup, and let φ be some continuous mapping of the space A_t into the interval $[0,1]$. To each element a of the semigroup A we assign the continuous mapping φ_a of the space A_t into the interval $[0,1]$:

$$\forall_{x \in A} \varphi_a(x) = \varphi(xa).$$

The set $J_\varphi = \{\varphi_a\} a \in A$ is a metric space with respect to the metric r:

$$r(\varphi_a, \varphi_b) = \max_{x \in A} |\varphi_a(x) - \varphi_b(x)|,$$

and the metric space J_φ is compact.

On the compactum J_φ we define multiplication:

$$\forall_{\varphi_a, \varphi_b \in J_\varphi} \varphi_a \cdot \varphi_b = \varphi_{ab}.$$

Then the mapping $f(a) = \varphi_a$ of the topological semigroup A_t onto J_φ will be a continuous homomorphism and, by Lemma 5, the semigroup J_φ is topological.

From all that has been said it follows that

Theorem 5. For every bicompact topological semigroup there exists a representation in a compact metric semigroup.

Sec. 9. We shall say that a topological semigroup A_t admits a sufficient system of representations in the class of topological semigroups Σ, if for any two elements $x, y \in A_t$ there exists a representation f of the topological semigroup A_t in a topological semigroup from the class Σ, for which $f(x) \neq f(y)$.

Theorem 6. A bicompact topological semigroup A_t admits a sufficient system of representations in the class of metrizable compact semigroups.

Sec. 10. From the results of Sec. 4 it follows directly that a compact metric semigroup is i-representable in the class of semigroups of continuous transformations of compacta. In turn, every compactum is homeomorphic to some subset of the Hilbert cube. Together with Theorem 6 this gives the following theorem:

Theorem 7. A bicompact topological semigroup A_t admits a sufficient system of representations in the class of semigroups of continuous transformations of compact subsets of the Hilbert cube.

Sec. 11. S. Ulam ([5]) proposed the following problem: does there exist a universal bicompact topological semigroup, i.e. a topological-

topological semigroup A_t such that every bicompact topological semigroup is continuously isomorphic to a subsemigroup of it?

From item 10 it follows immediately:

Theorem 8. *Every bicompact topological semigroup A_t is topologically isomorphic to some subsemigroup of the direct product of all semigroups of continuous transformations of compact subsets of the Hilbert cube.*

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Note: Figure translations are in progress. See original paper for figures.