



Soviet-era science, translated into English

ON THE DECISION PROBLEM

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.60321>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.11

MATHEMATICS

Yu. Sh. GUREVICH

ON THE DECISION PROBLEM

FOR THE PURE NARROW PREDICATE CALCULUS

(Presented by Academician A. I. Mal'cev on 26 V 1965)

1. In ⁽¹⁾ a summary is given of results on the reduction of the decision problem for satisfiability for the pure narrow predicate calculus. It is indicated there that, if one excludes classes of formulas of the form

$$\mathfrak{G}^a \mathfrak{G}^b \mathfrak{G}^n M(F_1, \dots, F_c),$$

containing not only unary predicates, where a, b, c are bounded in the given class and $b > 0$, then for any of the remaining classes of prenex formulas without free variables and with a fixed set of predicates the problem is solved.

Theorem. Let σ contain one binary and $(9 + k)$ unary predicates, where $2^k \geq m^*$ and m^* is the number of states of a universal Turing machine. The sets of refutable and finitely satisfiable σ -formulas of the form $\forall \exists \forall \& \mathfrak{G}^n$ are recursively inseparable.

The present paper is devoted to the proof of this theorem.

2. By a Turing machine we shall here mean the variant of such machines described in ⁽²⁾. M^* will further denote a universal Turing machine. Let X_1 (respectively X_2) be the set of those n such that M^* halts from the tape (respectively cycles), if in the initial situation

$$\overbrace{11 \dots 100 \dots}^{n+1}$$

M^* scans the leftmost cell. X_1 and X_2 are recursively inseparable.

3. In ⁽²⁾, for each Turing machine M there is constructed a formula $\alpha_M(x, x', y)$, containing binary predicates S, K , and H and a certain number of unary ones, such that M , scanning at the zero moment the

zero cell of the empty tape, eventually halts from the tape (respectively cycles) if and only if $\forall xy \alpha_M(x, x+1, y) \cdot Z0$ is unsatisfiable (respectively satisfiable by periodic predicates) in the domain of natural numbers. Here Zy is interpreted as: y is zero, and Syx as: at moment y the cell numbered x is nonempty. In α_{M^*} we replace $(Zy \supset \neg Syx')$ by $(Zy \supset (Syx' \supset Syx))$. We obtain the formula $\alpha(x, x', y)$.

Lemma 1. $n \in \overline{X}_1$ (respectively $n \in X_2$) if and only if $\forall xy \alpha(x, x+1, y) \cdot Z0 \cdot S0n \cdot \neg S0(n+1)$ is satisfiable (respectively satisfiable by periodic predicates) in the domain of natural numbers.

4. Let $\beta(x, x', y)$ be the conjunction of the following five formulas:

- 4.1. $\neg Ax \cdot \neg Ay \cdot Cxy \supset Dx'y.$
- 4.2. $Dyx \supset \neg Ax' \cdot Cyx'.$
- 4.3. $Ay \cdot Cyx \supset Ax'.$
- 4.4. $Ax \cdot Bx \cdot Zy \supset Syx \cdot \neg Syx'.$
- 4.5. $Bx \supset Bx'.$

Let

$$\begin{aligned} \varphi_n = & \forall x \mathfrak{G}x' \forall y (\alpha(x, x', y) \cdot \beta(x, x', y)) \ \& \\ & \& \mathfrak{G}x_0 x_1 \dots x_n [Ax_0 \cdot \bigwedge_{i>0} \neg Ax_i \cdot \bigwedge_{i<n} Cx_i x_{i+1} \cdot \\ & \cdot \bigwedge_{1+i \neq j} \neg Cx_i x_j \cdot Zx_n \cdot Bx_n]. \end{aligned}$$

Lemma 2. $\varphi = [\forall xy \alpha(x, x+1, y) \cdot Z0 \cdot S0n \cdot \neg S0(n+1)]$ is satisfiable (respectively, finitely satisfiable by periodic predicates) in the domain of natural numbers if and only if φ_n is satisfiable (respectively, finitely satisfiable).

Let us show how a model for φ_n is constructed if there exists a suitable model \mathfrak{M} for φ . Let

$$|\mathfrak{M}_i| = \{(i, k) \mid k = 0, 1, \dots\}$$

and let $(i, k) \leftrightarrow k$ be an isomorphism of \mathfrak{M}_i and \mathfrak{M} , $i = 0, \dots, n$.

For $i \neq j$ we put

$$\neg S(i, k)(j, l), \quad \neg K(i, k)(j, l), \quad \neg H(i, k)(j, l).$$

Further, we put:

$$\begin{aligned} A(i, k) & \text{ is equivalent to } i = k, \\ B(i, k) & \text{ is equivalent to } i = n, \\ C(i, k)(j, l) & \text{ is equivalent to } j = i + 1 \text{ and } k = l \leq i, \\ D(i, k)(j, l) & \text{ is equivalent to } j = i + 1 \text{ and } k = l + 1 \leq i. \end{aligned}$$

The pairs (i, k) , with the relations defined, form a model for φ_n . If, conversely, there is a model \mathfrak{M} for φ_n , then, without loss of generality, we may assume that $|\mathfrak{M}|$ consists of pairs (i, k) , where $i = 0, \dots, n$ and $k = 0, 1, \dots$. The element $(i, 0)$ plays the role of x_i from φ_n . By induction on i , $A(i, i)$ is proved. From $A(n, n)$ and $B(n, 0)$, by means of 4.5 and 4.4, one derives $S(n, 0)(n, n)$ and $\neg S(n, 0)(n, n + 1)$. The elements (n, k) , $k = 0, 1, \dots$, with the corresponding relations form a suitable model for φ .

5. Let \mathfrak{M} be a model for φ_n . Replace each element a of \mathfrak{M} by eleven new elements a^0, \dots, a^{10} . Let h be any unary predicate from φ_n ; if ha held (respectively, $\neg ha$), put ha^i (respectively, $\neg ha^i$), $i = 0, \dots, 10$. We now define on the set

$$\{a^i \mid a \in \mathfrak{M}, i = 0, \dots, 10\}$$

a new predicate Fxy (or simply xy):

$$5.1. \quad Cab \rightarrow \bigwedge_i (a^i b^0 . b^i a^2) . b^i a^{10}.$$

$$5.2. \quad Dab \rightarrow \bigwedge_i (a^i b^1 . b^i a^3).$$

5.3; 5.4; 5.5—analogously to 5.1 and 5.2, but for S , K , and H , respectively.

5.6. $a^i b^j$ holds only in the case when this follows from 5.1–5.5.

As a result we obtain a model for the formula ψ_n defined below.

6. Let $\gamma(x, x', y)$ be the conjunction of the following formulas:

$$6.1. \quad \bigvee_{i=0}^{10} f^i x . \bigwedge_{i \neq j} \neg (f^i x . f^j x).$$

$$6.2. \quad \bigwedge_{i < 10} (f^i x \sim f^{i+1} x').$$

$$6.3. \quad \bigwedge_{h, i < 10} f^i x \supset (hx \sim hx')$$

(here h ranges over the unary predicates from φ_n).

$$6.4. \quad \bigwedge_{i < 10} [f^i x . (f^0 y \vee f^2 y) \supset (xy \sim x'y)] . [f^{10} y . f^0 x \supset (yx \sim x'y)].$$

6.5–6.8. Formulas concerning D , S , K , and H , respectively, just as 6.4 concerns C .

$$6.9. \quad f^{10}x.f^1y.\neg Ax.\neg Ay.yx \subset x'y.$$

6.10 and thereafter—formulas concerning 4.2, ..., 4.5 and the conjuncts from $\alpha(x, x', y)$, just as 6.9 concerns 4.1.

Let

$$\begin{aligned} \psi_n = & \forall x \exists x' \forall y \gamma(x, x', y) \ \& \\ & \& \exists x_0 x_1 \dots x_n \left[\bigwedge_i f^0 x_i . A x_0 . \bigwedge_{i>0} \neg A x_i . \right. \\ & \left. \cdot \bigwedge_{i<n} x_i x_{i+1} . \bigwedge_{1+i \neq j} \neg x_i x_j . Z x_n . B x_n \right]. \end{aligned}$$

Lemma 3. φ_n is satisfiable (respectively, finitely satisfiable) if and only if ψ_n is satisfiable (respectively, finitely satisfiable).

7. ψ_n contains one binary predicate and $m^* + 17$ unary ones. The individual groups of unary predicates here satisfy a property similar to 6, 1, which makes it possible to reduce the necessary number of unary predicates to $9 + k$, where $2^k \geq m^*$.

Thus, the assertion of the theorem follows from Lemmas 1, 2, and 3.

I express my gratitude to Academician A. I. Mal' tsev for posing the problem.

Ural State University
named after A. M. Gorky

Received
21 V 1965

REFERENCES

1. V. F. Kostyrko, *Algebra and Logic*, **3**, no. 5-6, 45 (1964).
2. J. R. Buchi, *Math. Ann.*, **148**, 201 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.