SOME PROPERTIES OF QUASI-NUMBERS AND OF OPERATORS FROM QUASI-NUMBERS TO QUASI-NUMBERS

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Abstract

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SOME PROPERTIES OF QUASI-NUMBERS AND OF OPERATORS FROM QUASI-NUMBERS TO QUASI-NUMBERS

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1. We consider algorithms in the standard extension of the alphabet of FR-numbers Ψ_3 (see ([1]), p. 77).

We shall say that an algorithm φ is **consistent at the point** q_0 (q_0 is a quasi-number) if $!\varphi(q_0)$, and for every quasi-number q_1 such that $q_1 = q_0$, one has $!\varphi(q_1)$, and $\varphi(q_0)$, $\varphi(q_1)$ are equal quasi-numbers. Every algorithm in the standard extension of the alphabet Ψ_3 will be called a **quasi-function**. We shall say that a quasi-function φ is **defined at the point** q_0, if the algorithm φ is consistent at this point. The formulated definitions of a quasi-function and of the points of its definedness are analogous to the definitions of a pseudo-operator and of the points of its definedness adopted in ([2]).

By analogy with the definition of a constructive function, and also of an operator in ([2]), one can introduce the notion of an operator from quasi-numbers to quasi-numbers: a quasi-function φ is called an **operator from quasi-numbers to quasi-numbers** if, for every quasi-number q, from $!\varphi(q)$ there follows the consistency of the algorithm φ at the point q.

It is not difficult to show that for any operator from quasi-numbers to quasi-numbers its domain of definedness is either the empty set or coincides with the set of all quasi-numbers. This fact (in several other formulations) was discovered independently and earlier by G. E. Mints ([3]) and V. A. Shurygin. Thus, the notion of an operator from quasi-numbers to quasi-numbers with nonempty domain of definedness coincides with the notion of an everywhere-defined quasi-function.

Let φ be a quasi-function. We shall say that φ has a **constructive discontinuity at the point** q_0, if φ is defined at q_0 and there exist a sequence of quasi-numbers ω and a rational number $r > 0$ such that, for every n, the quasi-function φ is defined at the point $\omega(n)$, $|\omega(n) - q_0| < 2^{-n}$, and $|\varphi(\omega(n)) - \varphi(q_0)| \geq r$.

The main theorem of this section is the following.

Theorem 1. *A quasi-function cannot have a constructive discontinuity at any point.*

The formulation of this theorem is close to the formulation of the well-known theorem of A. A. Markov on the absence of constructive discontinuities for constructive functions ([4]). At the same time, the proof of Theorem 1 known to the author differs very substantially from the proof of the aforementioned theorem of A. A. Markov.

The proof of Theorem 1 is based on the following lemma, which is of independent interest.

Lemma 1. *There is no algorithm that transforms the record of any sequence of rational numbers into a quasi-number to which this sequence does not converge.*

The proof of this lemma was strongly influenced by certain ideas of G. S. Tseitin.

Remark 1. Using G. S. Tseitin' s theorem on the continuity of constructive functions, it is not difficult to show that for every everywhere de-

of a given constructive function φ, one can construct an everywhere defined quasifunction ψ such that, for every duplex x and quasinumber q, from the equality $x = q$ there follows the equality $\varphi(x) = \psi(q)$. This circumstance makes it possible to use the known results of I. D. Zaslavskii and G. S. Tseitin ([6,7]) for constructing quasifunctions with properties unusual for classical analysis: for example, to construct a discontinuous quasifunction unbounded on the segment $0\Delta 1$.

Remark 2. In connection with Theorem 1, the question of the continuity of quasifunctions naturally arises.

It is not difficult to show that there is no algorithm which, for every operator from quasinumbers to quasinumbers φ, finds a rational number $r > 0$ such that, for every quasinumber q, from the inequality $|q| < r$ there follows $|\varphi(q) - \varphi(0)| < 1$. Such an algorithm is already impossible in the class of linear quasifunctions*. Moreover, one can construct an everywhere defined quasifunction φ (though neither linear nor polygonal) for which there is no algorithm that transforms each quasinumber q into a rational number $r_q > 0$ such that, for any quasinumber q_1 satisfying the inequality $|q_1 - q| < r_q$, one has $|\varphi(q_1) - \varphi(q)| < 1$.

The question of the truth of a weaker formulation of the continuity theorem (given a rational ε, one is required to find a quasinumber $\delta > 0$ satisfying the condition of continuity for ε) apparently remains open.

Remark 3. By a method close to the proof of Theorem 1, one can prove the impossibility of an algorithm applicable to the record of every algorithm of type $(\mathfrak{n} \to \mathfrak{n})$ and transforming the record of every arithmetically complete algorithm into a quasinumber equal to 0, and of every arithmetically incomplete algorithm into a quasinumber equal to 1.

2. Theorem 2. *One can construct a sequence of quasinumbers ω such that,* *for every duplex x, one can indicate a natural number l for which the equality $x = \omega(l)$ holds.*

Thus, the set of all quasinumbers turns out to be countable in a certain sense.

Theorem 2 was found independently by G. S. Tseitin and the author.

Remark 4. Theorems analogous to Theorem 2 hold for certain other sets of quasinumbers, in particular for the set of all pairs of unequal quasinumbers.

An algorithm ρ of type $(\mathfrak{n} \to \mathfrak{n})$ will be called a **regulator of convergence of the second kind** of a sequence of rational numbers $\mathfrak{A}$, if

$$\mathfrak{T}n\,\forall mlk\,((m \geq n\&l, k > \rho(m)) \supset (|\mathfrak{A}(l) - \mathfrak{A}(k)| < 2^{-m})).$$

An algorithm ρ will be called a **quasiregulator of convergence of the second kind** of a sequence of rational numbers $\mathfrak{A}$, if it cannot fail to be a regulator of convergence of the second kind of this sequence.

A word C will be called an FL (**quasi-**FL)-number if C is a rational number, or C has the form $C = \{\mathfrak{A}\}_3 \diamond \{\rho\}_3$, where $\mathfrak{A}$ is a sequence of rational numbers and ρ is its regulator (quasiregulator) of convergence of the second kind.

Theorem 3. *One can construct an algorithm transforming the record of every sequence of FL (quasi-FL)-numbers into a duplex distinct from all members of this sequence.*

The proof of Theorem 3 is carried out by means of a slight modification of the usual diagonal method.

It is not difficult to show that there is no algorithm applicable to every

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* A quasifunction φ is called **linear** if there exist quasinumbers p_1 and p_2 such that, for every q, one has $\varphi(q) = p_1 q + p_2$.

to an FL (quasi-FL) number x, for which $\neg\neg(x = 0 \vee x = 1)$, indicating a true term of the disjunction $(x = 0 \vee x = 1)$.

From this circumstance the following theorem follows.

Theorem 4. *There is no algorithm that constructs, for every FL (quasi-FL) number, an FR-number equal to it.*

The following theorem follows directly from Theorems 2 and 3.

Theorem 5. *There is no algorithm that constructs, for every quasi-number, a quasi-FL-number equal to it.*

Theorem 6. *There is no algorithm that constructs, for every F-number, a quasi-FL-number equal to it.*

Theorem 6 follows directly from Theorem 5 and is a strengthening of the well-known result of G. S. Tseitin ([8]) on the impossibility of an algorithm that constructs, for every F-number, an FR-number equal to it.

Corollary. *There is no algorithm that transforms the notation of every convergent sequence of rational numbers into the notation of a convergence quasiregulator of the second kind for this sequence.*

Remark 5. Using Theorem 5, it is not difficult to prove the following assertion: there is no algorithm α that transforms the notation of every convergent sequence of rational numbers $\mathfrak{A}$ into the notation of a sequence of arithmetically complete algorithms such that there cannot fail to exist an n for which the algorithm $\langle\langle\alpha(;\mathfrak{A};)\rangle(n)\rangle$ is a convergence regulator of the sequence $\mathfrak{A}$.

Theorem 7. *One can construct a sequence of quasi-FL-numbers $\mathfrak{A}$ such that*

$$\forall nm(((m \geq n) \supset (|\mathfrak{A}(n) - \mathfrak{A}(m)| < 2^{-n}))),$$

and such that there is no duplex (and hence no quasi-FL-number) which is the limit of this sequence.

Theorem 8. *One can construct a sequence of quasi-numbers $\mathfrak{A}$ such that*

$$\forall nm(((m \geq n) \supset (|\mathfrak{A}(n) - \mathfrak{A}(m)| < 2^{-n}))),$$

and such that there is no duplex (and hence no quasi-number) which is the limit of this sequence.

Theorem 8 is a weakened variant of Theorem 7.

Thus, quasi-FL-numbers and quasi-numbers form incomplete number systems, and quasi-FL-numbers give an example of an effectively uncountable incomplete number system.

The proof of Theorem 7 is based on Theorem 4.2 ([6]). In the course of the proof it is established that, for every duplex x, one can indicate rational numbers r_1 and r_2 and a number n such that $r_1 < x < r_2$ and the terms of the sequence $\mathfrak{A}$ with indices greater than n do not fall into $r_1 \nabla r_2$.

Theorem 8 is supplemented by the following theorem.

Theorem 9. *There is no algorithm that transforms the notation of every sequence of quasi-numbers which cannot fail to converge to 0 or to 1 into a quasi-number to which this sequence converges.*

For the proof of Theorem 9 it is enough to consider the series

$$\sum_{n=0}^{\infty} |\alpha|(1 - |\alpha|)^n$$

for $0 \leq |\alpha| \leq 1$ and apply Theorem 1. From the proof of Theorem 9 the following theorem is easily seen.

Theorem 10. *There is no algorithm that transforms the notation of every self-convergent sequence of F-numbers into an F-number that is the limit of this sequence.*

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Note: Figure translations are in progress. See original paper for figures.