

ON THE QUESTION OF CONTROLLABILITY OF DYNAMIC SYSTEMS\

Yu. M.-L. KOSTYUKOVSKII

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.59906>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.941.7

CYBERNETICS AND CONTROL THEORY

Yu. M.-L. KOSTYUKOVSKII

ON THE QUESTION OF CONTROLLABILITY OF DYNAMIC SYSTEMS*

(Presented by Academician B. N. Petrov on 20 VII 1965)

The controllability criterion for a linear nonstationary dynamic system is usually formulated in terms of the fundamental matrix of solutions ⁽¹⁻⁸⁾. In the present paper a correspondence is established between known and newly obtained conditions of “output controllability” (see ^(7,8)). These conditions are formulated in terms of the given matrices corresponding to the equations of motion of the dynamic system under consideration. The motion of the dynamic system is given by the matrix equations:

$$\frac{dx}{dt} = A(t)x + \sum_{s=1}^r b^s u_s, \quad (1)$$

$$y = C(t)x, \quad (2)$$

where x is the position vector of the dynamic system in the vector space P^n ; y is a vector (the output of the dynamic system) belonging to the vector space Q^m , $m \leq n$; $u_s(t)$ is an arbitrary piecewise-continuous function of time (control action); $A(t)$, $b^s(t)$, $C(t)$ are variable matrices of types $n \times n$, $n \times 1$, $m \times n$; the elements of the matrices $A(t)$, $b^s(t)$, $C(t)$ have continuous derivatives up to order $(n - 1)$, inclusive.

Introduce the notation:

- a) $D(t)$ is a matrix of type $n \times n$, formed by augmenting the matrix $C(t)$ to any nonsingular matrix;

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}; \quad B = (b^1, b^2, \dots, b^r);$$

$$E(t) = D(t)A(t)D^{-1}(t) + \dot{D}(t)D^{-1}(t),$$

where D^{-1} is the matrix inverse to the matrix D ;

$$F(t) = D(t)B(t); \quad F^s(t) = D(t)b^s(t);$$

- b) a sequence of operators defined as follows: $q^0 = E_n$, where E_n is the identity matrix of type $n \times n$; $q^1 = p - E(t)$; $q^2 = q^1(q^1)$, ..., $q^i = q^1(q^{i-1})$, ...; $p = d/dt$ is the differentiation operator;
- c) $\Phi(t, t_0)$ is the fundamental matrix of solutions of the homogeneous system corresponding to equation (1);
- d) the output of the dynamic system

$$y(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^t H_y(t, \tau)u \, d\tau, \quad (3)$$

* Reported at the Moscow-wide seminar on the theory of multivariable control systems on 17 II 1965 at the Institute of Automation and Telemechanics.

where

$$H_y(t, \tau) = C(t)\Phi(t, \tau)B(\tau); \quad (4)$$

$h_j(t, \tau)$ is the j -th column of the matrix $H_y(t, \tau)$; $j = 1, \dots, r$;

- d) by $\lambda^+ = [\lambda', 0]$ * is denoted a vector belonging to the direct product $Q^m \times T^{n-m}$, where λ is an arbitrary nonzero column vector belonging to Q^m ; 0 is the zero row vector belonging to the vector space T^{n-m} .

Theorem. The condition

$$H'_y(t_0 + T, t)\lambda \neq 0, \quad t_0 \leq t \leq t_0 + T, \quad (5)$$

is equivalent to the inequalities

$$(\lambda^+, q^{kF^{s(t)}} 0, \quad t_0 \leq t \leq t_0 + T, \quad (6)$$

for at least one of k, s ; $k = 0, 1, \dots, n-1$; $s = 1, \dots, r$.

In exactly the same way, the condition

$$h'_s(t_0 + T, t)\lambda \neq 0, \quad t_0 \leq t \leq t_0 + T, \quad (7)$$

is equivalent to the inequalities:

$$(\lambda^+, q^{kF^{s(t)}} 0, \quad t_0 \leq t \leq t_0 + T, \quad (8)$$

for all $s = 1, \dots, r$ and for at least one of $k = 0, 1, \dots, n - 1$.

Proof. Suppose that condition (5) is satisfied. Then the dynamic system is completely controllable at the output (8). Consider the system

$$dx/dt = A(t)x + B(t)u, \quad (9)$$

$$z = D(t)x, \quad (10)$$

where $z = (z_1, \dots, z_n)$; $z \in R^n \supset Q^m$.

Let e^1, \dots, e^n be an orthonormal basis in R^n , where

$$e^1 = (1, 0, \dots, 0); e^2 = (0, 1, \dots, 0); \dots; e^n = (0, 0, \dots, 1). \quad (11)$$

Then the increment

$$\Delta z_i(t_0 + T) = - \int_{t_0}^{t_0+T} \sum_{s=1}^r (\psi^i(\tau), F^s(\tau)) \Delta u_s d\tau, \quad (12)$$

where the functions $\psi^i(t)$, $i = 1, \dots, n$, satisfy the adjoint system

$$d\psi^i/dt = -E'(t)\psi^i, \quad i = 1, \dots, n, \quad (13)$$

$$\psi^i(t_0 + T) = -e^i \quad (14)$$

(see (9)).

We shall use the method of reverse motion. In order not to introduce new notation, we shall assume that $z(t_0) = z(0)$ is the prescribed final position. As the matrices $E(t)$, $F(t)$, one should take the matrices $E(2t_0+T-t)$, $F(2t_0+T-t)$. Let also $u(t) \equiv 0$.

It is easy to see that

$$(\psi^i(t), F^s(t)) \neq 0 \quad \text{on } t_0 \leq t \leq t_0 + T \quad (15)$$

for all $i = 1, \dots, m \leq n$ and for at least one of $s = 1, \dots, r$. If this were not so, then $\Delta z_i(t_0 + T) = 0$, $i = 1, \dots, m \leq n$, and this would contradict the supposition that the dynamic system is completely controllable at the output.

The identities

$$(\psi^i(t), F^s(t)) \equiv 0 \quad \text{on } t_0 \leq t \leq t_0 + T \quad (16)$$

* G' is the transposed matrix (or vector) G .

for all i, s ; $i = 1, \dots, m \leq n$; $s = 1, \dots, r$, and the identities

$$(e^i, q^{kF^{s(t)}}) = 0 \quad \text{for } t_0 \leq t \leq t_0 + T \quad (17)$$

for all i, k, s ; $i = 1, \dots, m \leq n$; $k = 0, 1, \dots, n-1$; $s = 1, \dots, r$, are equivalent conditions (see (10)).

Then, on the basis of (15),

$$(e^i, q^{kF^{s(t)}}) \neq 0 \quad (18)$$

for all $i = 1, 2, \dots, m \leq n$, and for at least one of k, s ; $k = 0, 1, \dots, n-1$; $s = 1, \dots, r$.

Since e^i , $i = 1, \dots, n$, is an orthonormal basis, (6) is also satisfied.

It is shown analogously that (8) implies (7).

We shall now prove the theorem in the reverse direction.

Following the terminology of ⁽¹⁰⁾, we shall call the dynamic system (1)–(2) weakly output-invariant with respect to $u_s(t)$ at the time $t = t_0 + T$, if the output of the dynamic system at this instant of time does not depend on $u_s(t)$.

The following assertion is true: in order that the dynamic system (1)–(2) be weakly output-invariant with respect to $u_s(t)$ at the time $t = t_0 + T$, it is necessary and sufficient that the condition

$$h'_s(t_0 + T, \tau) \lambda \equiv 0, \quad t_0 \leq t \leq t_0 + T \quad (19)$$

be satisfied.

Indeed, if the dynamic system is weakly output-invariant with respect to $u_s(t)$ at the time $t = t_0 + T$, then the scalar product of the vectors

$$V_{u_s} = (\lambda', y(t_0 + T) - C(t_0 + T)\Phi(t_0 + T, t_0)x_0) \equiv \int_{t_0}^{t_0 + T} h'_s(t_0 + T, \tau) \lambda u_s d\tau \quad (20)$$

is equal to zero for any $u_s(t)$.

If (19) were not satisfied, then, by virtue of the continuity of $h'_s(t_0 + T, t)$ in t , there would be a sufficiently small interval $[\tau_s, \tau'_s]$ such that

$$h'_s(t_0 + T, t(\lambda)) \neq 0 \quad \text{for all } t \in [\tau_s, \tau'_s].$$

Choose the control function

$$u_s^* = \begin{cases} N, & t \in [\tau_s, \tau'_s], \\ 0, & t \notin [\tau_s, \tau'_s], \end{cases} \quad (21)$$

where N is some number.

Substituting (21) into (20), as a result we obtain $V_{u_s^*}^* \neq 0$, which contradicts the assumption of invariance of the dynamic system. The assertion is proved.

Next, we note that

$$\text{either } h'_s(t_0 + T, t)\lambda \equiv 0, \quad t_0 \leq t \leq t_0 + T, \quad (22)$$

$$\text{or } h'_s(t_0 + T, t)\lambda \neq 0, \quad t_0 \leq t \leq t_0 + T. \quad (23)$$

Suppose that (8) is satisfied; then

$$(\psi^j(t), F^s(t)) \neq 0, \quad t_0 \leq t \leq t_0 + T,$$

for all $j = 1, \dots, m$ (see the proof of (18)).

Consequently, the dynamic system (1)–(2) is not weakly output-invariant with respect to $u_s(t)$ at the time $t_0 + T$, and therefore, by virtue of the alternative character of conditions (22), (23), (23) holds. It is shown analogously that if (6) is satisfied, relation (5) is true.

Remark 1. If one uses the duality principle (see ^(1,8)) and the result of the theorem proved, it is not difficult to formulate an observability criterion in terms of the given matrices $A(t), C(t)$.

Remark 2. In view of (17), (18), the following assertion is valid: under the conditions of applicability of the theorem proved, linear nonstationary dynamical systems are divided into: a) completely controllable dynamical systems possessing property (6); b) dynamical systems strongly invariant at the output with respect to $u_1(t), \dots, u_r(t)$ on the time interval $[t_0, t_0 + T]$, for which

$$(\lambda^+, q^k F^s(t)) \equiv 0, \quad t_0 \leq t \leq t_0 + T,$$

for all $k, s; k = 0, 1, \dots, n - 1; s = 1, \dots, r$ ⁽¹⁰⁾.

Moscow Institute
of Electronic Machine Building

Received
14 VII 1965

CITED LITERATURE

- ¹ R. E. Kalman, Proc. First IFAC Congress, 2, Acad. Sci. USSR Press, 1961, p. 521.
- ² F. M. Kirillova, PMM, 25, No. 3, 433 (1961).
- ³ J. Kurzweil, Avtomatika i telemekh., 22, 688 (1961).
- ⁴ L. Markus, E. B. Lee, J. Basic Eng. (Trans. ASME), 83D (1961).
- ⁵ E. A. Lidskii, PMM, 25, No. 5, 824 (1961).
- ⁶ N. N. Krasovskii, PMM, 23, No. 4, 625 (1959).
- ⁷ G. I. Bertram, P. I. Sarachik, Proc. First IFAC Congress, 2, Acad. Sci. USSR Press, 1961, p. 356.
- ⁸ E. Kreindler, P. E. Sarachik, J. IEEE, Trans. Automat. Control., AC-9, No. 2, 129 (1964).
- ⁹ L. I. Rozonoer, Avtomatika i telemekh., 20, No. 10, 1320 (1959).
- ¹⁰ L. I. Rozonoer, Avtomatika i telemekh., 24, No. 6, 744 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.