POSITIVE DEFINITE FUNCTIONS ON ALGEBRAIC NILPOTENT GROUPS OVER A DISCRETE FIELD

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Abstract

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POSITIVE DEFINITE FUNCTIONS ON ALGEBRAIC NILPOTENT GROUPS OVER A DISCRETE FIELD

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Let G be a discrete group. Denote by $M(G)$ the set of positive definite functions φ on G, constant on conjugacy classes and such that $\varphi(e) = 1$ (where e is the identity of G). It is known that there is a one-to-one correspondence between the extreme points of $M(G)$ and factor representations of type I_n and II_1.

The first complete description of the set $M(G)$ for a group not belonging to type I was given by Thoma ([1,2]) for the group of affine transformations of a one-dimensional space over the field of rational numbers and for the infinite symmetric group. In the work of A. A. Kirillov ([3]) this problem was solved for matrix groups of arbitrary order.

The present note is devoted to the case where G is a nilpotent linear algebraic group over a field k of characteristic zero.

Theorem. *Let H be an algebraic normal divisor of G; C the center of the factor group G/H; p the natural projection of G onto G/H. Denote by π a character of C which is not equal to 1 on any one-parameter subgroup of C. Then the function $\varphi(g)$, given by the formula*

$$\varphi(g) = \begin{cases} \pi(p(g)), & \text{if } p(g) \in C, \\ 0, & \text{if } p(g) \notin C, \end{cases}$$

is an extreme point of the set $M(G)$. Conversely, every extreme point of $M(G)$ can be obtained by such a construction.

Proof. First consider the group S_n corresponding to the Lie algebra $\mathcal{S}_n$ with basis $Y, X_1, \dots, X_n$ and commutation relations

$$[Y, X_i] = X_{i+1} \quad (i = 1, \dots, n-1); \qquad [Y, X_n] = 0 \quad \text{and} \quad [X_i, X_j] = 0.$$

Direct computations show that the center C of this group consists of elements of the form $c = \exp tX_n$, where exp is the canonical mapping of the algebra onto the group. The elements cg and g, where $c \in C,\ g \notin C$, are conjugate.

Lemma 1. *If a function φ on S_n belongs to $M(S_n)$, then for any $g \notin C$ the function $\varphi(c) - |\varphi(g)|^2$ is positive definite on C.*

The proof is obtained by considering the set $c_1, \dots, c_n \in C$ and $g, \dots, g$ (n times).

Let G be a nilpotent algebraic linear group over a field of characteristic 0, and let $\mathcal{G}$ be the corresponding Lie algebra. Then we have

$$\mathcal{G} = Y_n \supset Y_{n-1} \supset \cdots \supset Y_1 = Z,$$

where the Y_i are ideals such that $[\mathcal{G}, Y_i] \subset Y_{i-1}$, and Z is the center.

Lemma 2. *For any $y \in G$ ($y \notin Z$) there exists an algebra isomorphic to $\mathcal{S}_n$ which contains y.*

Proof. Let Y_m be an ideal such that $[y, Y_i] = 0$ for $i < m$, but $[y, Y_m] \neq 0$; then there exists an element $t \in Y_m$ such that $[t, y] = y_1 \neq 0$. Let $[t, y_1] = y_2 \neq 0, \dots, [t, y_{l-1}] = y_l \neq 0$, but $[t, y_l] = 0$. We shall prove that $[y_i, y_j] = 0$ for all i and j. Indeed, $[y_i, y_j] =$

$$= [[t, y_{i-1}]y_j] = -[[y_{i-1}, y_j], t] - [[y_j, t], y_{i-1}] = -[[y_{i-1}, y_j], t] - [y_{i-1}, y_{i+1}],$$

and by induction on i we obtain the assertion of the lemma.

Now take an arbitrary extreme function φ from $M(G)$. From the work of Thoma ([4]) it follows that the restriction of φ to the center is a character of the center.

Lemma 3. *Suppose that the restriction of φ to the center does not degenerate on any one-parameter subgroup of the center (i.e., does not become 1 on it). Then φ is concentrated on the center.*

Proof. We shall show by induction that the restriction of φ to $\exp Y_k$ is concentrated on $\exp Z$. Suppose that we have already proved this for Y_{m-1}, and prove it for Y_m. Take an arbitrary element $y \in Y_m$ and consider φ on $\exp \mathcal{S}_n$ ($\mathcal{S}_n$ is constructed by Lemma 2). If $y_l \in Z$, then the restriction of φ to the center C of the group S_n will be a nondegenerate character; if $y_l \notin Z$, then this restriction will be a function concentrated at e. In both of these cases the function $\varphi(c) - d$ (d is a positive constant) on C cannot be positive definite. Using Lemma 1, we obtain

$$\varphi(\exp y) = 0.$$

Conversely, it is not difficult to check that a nondegenerate character of the center, extended trivially to the whole group, is an extreme function. Indeed, if $\varphi(g) = \frac{1}{2}(\varphi_1(g) + \varphi_2(g))$, then on the center $\varphi_1 = \varphi_2 = \varphi$. But then, by Lemma 3, φ_1 and φ_2 are concentrated on the center, i.e. φ is extreme.

Now take an arbitrary extreme function φ. Let J be a maximal ideal for which the restriction of φ to $\exp J$ becomes 1. Straightforward computations show that the function φ, considered as a function on $G/\exp J$, no longer degenerates on any one-parameter subgroup of the center. Therefore, by Lemma 3, φ has the required form.

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Note: Figure translations are in progress. See original paper for figures.