Reports of the Academy of Sciences of the USSR

1966

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1966. Volume 170, No. 5

MATHEMATICS

Yu. S. KOLESOV

ON ONE CRITERION FOR THE EXISTENCE OF PERIODIC SOLUTIONS OF PARABOLIC EQUATIONS

(Presented by Academician I. G. Petrovsky on 10 I 1966)

1. Let Ω be a bounded open domain of the n-dimensional space E_n of points $x = \{x_1, \dots, x_n\}$, belonging to the class $A^{(2,\lambda)}$ (see ([1])). In the domain Ω consider the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i,k=1}^{n} a_{ik}(t,x,u)\,\frac{\partial^2 u}{\partial x_i \partial x_k} = f\left(t, x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right), \qquad (1)$$

where $a_{ik}(t,x,u) = a_{ki}(t,x,u)$,

$$\sum_{i,k=1}^{n} a_{ik}(t,x,u)\xi_i\xi_k \geq \gamma(R)\sum_{i=1}^{n}\xi_i^2 \qquad (\gamma(R) > 0)$$

for $-\infty < t < \infty,\ x \in \overline{\Omega}$ ($\overline{\Omega} = \Omega + \Gamma;$ Γ is the boundary of the domain Ω), $|u| \geq R$ (R is an arbitrary nonnegative number).

We shall assume that the functions $a_{ik}(t,x,u)$ and $f(t,x,u,v_1,\dots,v_n)$ are ω-periodic in t, satisfy a local Hölder condition in the variables t and x, and, in the variables $u, v_1, \dots, v_n$, a local Lipschitz condition.

Finally, we shall assume that

$$|f(t,x,u,v_1,\dots,v_n)| \leq C_1(R) + C_2(R)\sum_{i=1}^{n}|v_i|^{2-\varepsilon_i(R)},$$

where $0 < \varepsilon_i(R) \leq 2,\ -\infty < t < \infty,\ x \in \overline{\Omega},\ |u| \leq R.$

We shall be interested in the question of the existence for equation (1) of ω-periodic in t solutions $u(t,x)$ $(-\infty < t < \infty,\ x \in \Omega)$, satisfying the boundary condition

$$u(t,x) = 0 \qquad (x \in \Gamma). \tag{2}$$

To study this problem we shall use a general method (see ([2])), which consists in the fact that equation (1), together with the boundary condition (2), is written in the form of an operator equation

$$du/dt + \mathcal{L}(t,u)u = f(t,u) \tag{3}$$

in a suitably chosen functional space E (in the theorem proved below, equation (3) is considered in the space C_0 of functions continuous on $\overline{\Omega}$ and vanishing on Γ), then to each initial condition $u(0) = u_0$ there is assigned the value Tu_0 of the solution of equation (3) at $t = \omega$. It turns out that the operator $U = \mathcal{L}^{\alpha}(0,0)T\mathcal{L}^{-\alpha}(0,0)$ is completely continuous in the space C_0 for some $\alpha \in (1/2, 1)$. Clearly, each fixed point u_0 of the operator U determines the initial value $\mathcal{L}^{-\alpha}(0,0)u_0$ of an ω-periodic solution of equation (3).

2. Let there exist functions $\psi_1(t,x)$ and $\psi_2(t,x)$, smooth in the closed domain $\overline{\Omega}$, such that for $0 \le t \le \omega$, $x \in \overline{\Omega}$ the inequalities

$$\frac{\partial \psi_1}{\partial t} - \sum_{i,k=1}^{n} a_{ik}(t,x,\psi_1)\frac{\partial^2 \psi_1}{\partial x_i \partial x_k} \ge f\left(t, x, \psi_1, \frac{\partial \psi_1}{\partial x_1}, \dots, \frac{\partial \psi_1}{\partial x_n}\right), \tag{4}$$

$$\psi_1(0,x) \ge \psi_1(\omega,x), \tag{5}$$

$$\frac{\partial \psi_2}{\partial t} - \sum_{i,k=1}^{n} a_{ik}(t,x,\psi_2)\frac{\partial^2 \psi_2}{\partial x_i \partial x_k} \le f\left(t, x, \psi_2, \frac{\partial \psi_2}{\partial x_1}, \dots, \frac{\partial \psi_2}{\partial x_n}\right), \tag{6}$$

$$\psi_2(0,x) \le \psi_2(\omega,x), \tag{7}$$

and for $0 \le t \le \omega$, $x \in \Gamma$

$$\psi_1(t,x) \ge 0, \qquad \psi_2(t,x) \le 0. \tag{8}$$

We shall assume that for $t = 0$, $x \in \Omega$ the functions $\psi_1(t,x)$ and $\psi_2(t,x)$ are related by

$$\psi_1(0,x) \ge \psi_2(0,x). \tag{9}$$

Theorem. *Let there exist functions $\psi_1(t,x)$ and $\psi_2(t,x)$ satisfying conditions (4)–(9). Then equation (1) has at least one ω-periodic solution $u^*(t,x)$ which satisfies the boundary condition (2) and the inequalities*

$$\psi_2(t,x) \leq u^*(t,x) \leq \psi_1(t,x) \qquad (0 \leq t \leq \omega,\ x \in \Omega).$$

If the solutions of equation (3) are nonlocally extendable, then for the proof one must consider the set S_1 of elements $u_0 \in C_0$ satisfying the inequalities

$$\psi_2(0,x) \leq \mathcal{L}^{-\alpha}(0,0)u_0 \leq \psi_1(0,x).$$

This set is convex and closed. From the theorem on differential inequalities (see, for example, (3)) it follows that

$$\psi_2(0,x) \leq T\mathcal{L}^{-\alpha}(0,0)u_0 \leq \psi_1(0,x).$$

The last inequalities are equivalent to the inequalities

$$\psi_2(0,x) \leq \mathcal{L}^{-\alpha}(0,0)Uu_0 \leq \psi_1(0,x).$$

Consequently, $Uu_0 \in S_1$ for $u_0 \in S_1$. It can be shown (using the method developed in (4)) that the set $S_2 = US_1$ is compact in the space C_0. Therefore the assertion of the theorem follows from the Schauder principle.

The proof is somewhat complicated if the solutions of equation (3) can, in finite time, "go to infinity."

3. **Corollary 1.** *Let*

$$f(t,x,R_1,0,\dots,0) \leq 0, \qquad f(t,x,-R_2,0,\dots,0) \geq 0,$$

where $0 \leq t \leq \omega$; $x \in \Omega$; R_1, R_2 are some nonnegative numbers. Then equation (1) has at least one ω-periodic solution $u^*(t,x)$ satisfying the inequalities

$$-R_2 \leq u^*(t,x) \leq R_1.$$

Corollary 2. *Suppose that the operator $\mathcal{L}(t,u)$ does not depend on t. Let*

$$\overline{f}(x,u,v_1,\dots,v_n) \geq \max_{0\leq t\leq\omega} f(t,x,u,v_1,\dots,v_n),$$

$$\underline{f}(x,u,v_1,\dots,v_n) \leq \min_{0\leq t\leq\omega} f(t,x,u,v_1,\dots,v_n).$$

Assume that the equations

$$\mathcal{L}(u)u = \overline{f}\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right),$$

$$\mathcal{L}(u)u = \underline{f}\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$$

have such solutions $\overline{u}(x)$, $\underline{u}(x)$ that

$$\overline{u}(x) \geqslant \underline{u}(x) \qquad (x \in \Omega);$$

$$\overline{u}(x) \geqslant 0, \qquad \underline{u}(x) \leqslant 0 \qquad (x \in \Gamma).$$

Then equation (1) has at least one ω-periodic solution $u^*(t, x)$, satisfying the boundary condition (2) and the inequalities

$$\underline{u}(x) \leqslant u^*(t, x) \leqslant \overline{u}(x).$$

Corollaries 1 and 2 contain, as special cases, existence theorems for periodic solutions of equation (1) obtained by I. I. Shmulev ([5]) and G. Prodi ([6]) by other methods.

Received
4 I 1966

CITED LITERATURE

[1] C. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.
[2] M. A. Krasnosel' skii, P. E. Sobolevskii, *Materials for the Soviet-American Symposium on Partial Differential Equations*, Novosibirsk, 1963, p. 3.
[3] S. Kaplan, Comm. Pure and Appl. Math., **16**, No. 3, 305 (1963).
[4] P. E. Sobolevskii, Tr. Moscow Math. Soc., **10**, 298 (1961).
[5] I. I. Shmulev, DAN, **141**, No. 6, 1313 (1961).
[6] G. Prodi, Atti IV Congr. Unione mat. ital., **2**, 1953, p. 193.

Note: Figure translations are in progress. See original paper for figures.