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ONE-SIDED ESTIMATES UNDER CONDITIONS OF ASYMPTOTIC STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS

MATHEMATICS

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Abstract

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MATHEMATICS

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ONE-SIDED ESTIMATES UNDER CONDITIONS OF ASYMPTOTIC STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS

(Presented by Academician I. N. Vekua, May 24, 1965)

We consider the question of asymptotic stability in the sense of Lyapunov of the zero solution of the differential equation

$$dx/dt = A(t)x + f(t, x) \tag{1}$$

in a Banach space E.

1. Auxiliary theorem. By (l, x) we shall denote the values of the linear functional $l \in E^*$ (E^* is the space conjugate to E) on the element x. We shall assume that the norm in E is Gateaux differentiable:

$$\lim_{\lambda \to 0} \frac{\|x + \lambda h\| - \|x\|}{\lambda} = (\Gamma x, h),$$

where $\Gamma x = \operatorname{grad} \|x\|$.

It is easy to verify that the operator Γ maps E into E^*, and moreover (see, for example, ([1]))

$$(\Gamma x, x) = \|x\|, \qquad \Gamma(\alpha x) = \Gamma x \quad (\alpha > 0). \tag{2}$$

If we put $E = L_p(G)$, then

$$\Gamma x = x|x|^{p-2}/\|x\|^{p-1}.$$

Consider a function $\gamma(t)$ which, for all $t \geq 0$ and $x \in D$, gives the estimate

$$(\Gamma x, A(t)x) \leq \gamma(t)\|x\|, \tag{3}$$

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where D is an everywhere dense domain of definition of the operator $A(t)$.

Introduce the notation

$$\Omega_\gamma = \lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma(s)\,ds, \qquad \Omega = \inf \Omega_\gamma,$$

where the lower bound is taken over all functions $\gamma(t)$ of the indicated class. The number Ω will be called the **central characteristic exponent** (cf. ([2])).

We shall assume that the operator $f(t,x)$ satisfies the condition

$$(\Gamma x, f(t,x)) \le \delta \|x\|. \tag{4}$$

Theorem 1. *For any $\varepsilon > 0$ there exists a sufficiently small $\delta > 0$ such that every solution $x(t)$ of equation (1) with an operator satisfying condition (4) admits the estimate*

$$\|x(t)\| \le \|x(0)\| C_\varepsilon \exp[(\Omega + 2\varepsilon)t],$$

where C_ε depends only on ε.

Proof. Let $v(t) = \|x(t)\|$. Then

$$dv(t)/dt = (\Gamma x, dx/dt) = (\Gamma x, A(t)x) + (\Gamma x, f(t,x)) \le \gamma(t)\|x\| + \delta\|x\|,$$

i.e.,

$$dv(t)/dt \le [\gamma(t) + \delta] v(t).$$

Applying the theorem on differential inequalities, from this we obtain

$$\|x(t)\| \le x(0) \exp\left[\int_0^t (\gamma(s) + \delta)\,ds\right]. \tag{5}$$

From the definition of Ω we have: for a given $\varepsilon > 0$ there exists a function $\gamma(t)$ such that (3) is satisfied and at the same time

$$\Omega_\gamma < \Omega + \varepsilon,$$

i.e.,

$$\int_0^t \gamma(s)\,ds \le C'_\varepsilon + (\Omega + \varepsilon)t.$$

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Then, choosing $\delta \le \varepsilon$ and taking this last inequality into account, from (5) we obtain

$$\|x(t)\| \le \|x(0)\| C_\varepsilon \exp[(\Omega + 2\varepsilon)t].$$

The theorem is proved.

2. Main theorem. Using Theorem 1, we shall prove the following theorem, which gives conditions for the asymptotic stability of the zero solution of equation (1).

Theorem 2. *Let the operator $f(t,x)$, for small $\|x\|$ and $t \in [0,\infty)$, satisfy the condition*

$$(\Gamma x, f(t,x)) \le \mathcal{L}\|x\|^{1+\alpha} \quad (\alpha > 0). \tag{6}$$

Let $\Omega < 0$.

Then the trivial solution of equation (1) is asymptotically stable.

Proof. Choose $\lambda > 0$ so that $\Omega_1 = \Omega + \lambda < 0$, and make the substitution

$$x(t) = \exp(-\lambda t) y(t).$$

Then

$$dy/dt = [A(t) + \lambda I]y + g(t,y), \tag{7}$$

where $g(t,y) = \exp(\lambda t) f[t, \exp(-\lambda t)y]$.

Taking into account (2) and condition (6), we have

$$(\Gamma y, g(t,y)) = e^{\lambda t}(\Gamma y, f[t, e^{-\lambda t}y]) =$$

$$= e^{\lambda t}(\Gamma[e^{-\lambda t}y], f[t, e^{-\lambda t}y]) \le e^{\lambda t}\mathcal{L}\|e^{-\lambda t}y\|^{1+\alpha},$$

$$(\Gamma y, g(t,y)) \le \mathcal{L}e^{-\lambda\alpha t}\|y\|^{1+\alpha}. \tag{8}$$

From (2) and (3) it follows that

$$(\Gamma y, [A(t) + \lambda I]y) = (\Gamma y, A(t)y) + \lambda(\Gamma y, y) =$$

$$= (\Gamma y, A(t)y) + \lambda\|y\| \ge [\gamma(t) + \lambda]\|y\|,$$

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$$(\Gamma y, [A(t) + \lambda I]y) \geq [\gamma(t) + \lambda]\|y\|.$$

Consequently, the central exponent of the operator $A(t) + \lambda I$ is equal to $\Omega_1 = \Omega + \lambda$, and therefore, choosing $\varepsilon > 0$ so that $\Omega_1 + 2\varepsilon < 0$, one can choose a function $\gamma(t)$ (according to the definition of Ω) satisfying inequality (3) and $\Omega < \Omega_1 + \varepsilon < 0$.

Let $\delta > 0$ be such that $\delta < \varepsilon$. Choose the initial time $t_0 > 0$ so large that, for $t \geq t_0$ and small $\|y\|$, (8) gives

$$(\Gamma y, g(t, y)) \leq \delta\|y\|.$$

Thus the operator $g(t, y)$ satisfies the conditions of Theorem 1; consequently,

$$\|y(t)\| \leq \|y(t_0)\|C_\varepsilon \exp[(\Omega_1 + 2\varepsilon)t].$$

Since $\Omega_1 + 2\varepsilon < 0$, the zero solution of equation (7) is asymptotically stable, and hence, a fortiori, the solution of equation (1) is asymptotically stable. The theorem is proved.

3. Examples

1) In the case when $E = E^n$, where $A(t)$ is an n-dimensional Euclidean space, an analogous question was considered in paper ([2]). However, even in this case the assertion of Theorem 2 is new, since, in contrast to ([2]), here the nonlinear term is subject to a one-sided estimate.
2) As is known (see ([3], [4])), if the operator $A(t)$ satisfies condition (3) (and if $\gamma(t) = \gamma^0 < 0$), then it is the infinitesimal generator of a strongly continuous semigroup, i.e. it is an "abstract elliptic operator." This fact means that, as a second example of equation (1), one may take a parabolic equation (of second and higher orders).

In conclusion, we note that a countable number of differential equations, integro-differential equations, etc., may also serve as examples of equation (1).

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Note: Figure translations are in progress. See original paper for figures.

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