THE INVERSE PROBLEM FROM SCATTERING DATA FOR A SYSTEM OF DIRAC EQUATIONS OF ORDER \(2n\)

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Abstract

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THE INVERSE PROBLEM FROM SCATTERING DATA FOR A SYSTEM OF DIRAC EQUATIONS OF ORDER $2n$

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Let

$$B_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad I_n = \begin{pmatrix} 0 \dots 0 & 1 \\ 0 \dots 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 \dots 0 & 0 \end{pmatrix}, \qquad C_n = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix},$$

where I_n and E_n are matrices of order n, E_n is the identity matrix; $Q(x)$ is a Hermitian matrix-valued function of order $2n$; y is a column vector with $2n$ components. The equation

$$B_n y' + mC_n y + Q(x)y = \lambda y, \qquad 0 \le x < \infty, \tag{1}$$

will be called a system of Dirac equations of order $2n$ (here m is the mass). In this paper the inverse problem of scattering theory (for the statement of the problem, see item 3) is solved for system (1). In the case $n = 1$ this problem was solved in (^{1}), in the case of one Schrödinger equation—in (^{2}), and for a system of Schrödinger equations—in (^{3}).

1. Canonical form of the Dirac system

If in system (1)

$$Q(x) = \begin{pmatrix} P & \Omega \\ \Omega^* & -P \end{pmatrix}, \tag{2}$$

where P and Ω are matrices of order n, $P = P^*$, and the matrix Ω is symmetric with respect to the second diagonal, then system (1) is called canonical. In what follows we shall consider only the canonical Dirac system, since in other cases the inverse problem of scattering theory is solved nonuniquely. Any other system of Dirac equations, by means of orthogonal transformations commuting with the matrix B_n, can be reduced to a canonical system. Thus, in what follows we shall consider system (1) under the assumption that $Q(x)$ has the form indicated in (2), and for each x from $[0, \infty)$

$$\|P(x)\| \leq C/(1+x)^{1+\varepsilon}, \qquad \|\Omega(x)\| \leq C/(1+x)^{2+\varepsilon}, \tag{3}$$

where C and ε are positive constants.

2. Transformation operator with a condition at infinity

Theorem 1. If the matrix-valued function $Q(x)$ satisfies condition (3), then for every λ for which $\operatorname{Im}\lambda \geq 0$, system (1) has a matrix solution $f(x,\lambda)$, which as $x \to \infty$ tends to

$$f_0(x,\lambda) = \begin{pmatrix} E_n \sqrt{\dfrac{\lambda+m}{\lambda-m}} \\ -iI_n \end{pmatrix} e^{i\lambda\sqrt{1-m^2/\lambda^2}}, \tag{4}$$

and there exists a matrix function $K(x,t)$ of order $2n$ such that

$$f(x,\lambda) = f_0(x,\lambda) + \int_0^\infty K(x,t) f_0(t,\lambda)\, dt, \tag{5}$$

where the norm

$$\|K(x,t)\| \leq C/(1+x)(1+t)^{1+\varepsilon}. \tag{6}$$

Further, if $Q(x)$ has a derivative, then $K(x,t)$ satisfies the equation

$$\left\{B_n \frac{\partial}{\partial x} + mC_n + Q(x)\right\} K(x,t) = -\frac{\partial}{\partial t} K(x,t) B_n + mK(x,t) C_n \tag{7}$$

and the conditions

$$B_{nK}(x,x) - K(x,x)B_n = Q(x), \tag{8}$$

$$K(x,t) \to 0 \quad \text{as } t \to \infty. \tag{9}$$

Conversely, if $K(x,t)$ satisfies equation (7) and conditions (6), (8), (9), then the matrix function $f(x,\lambda)$ defined by formula (5) satisfies equation (1) with a matrix function $Q(x)$ satisfying condition (3).

3. Parseval' s Equality and the Inverse Problem of Scattering Theory

We adjoin to equation (1) the boundary conditions

$$y_1(0) = \dots = y_n(0) = 0, \tag{10}$$

where $y_1, \dots, y_n$ are the first n components of the vector-function $y(x)$. Let $\varphi(x, \lambda)$ be the matrix solution of equation (1) with the initial condition

$$\varphi(0, \lambda) = \begin{pmatrix} 0 \\ -E_n \end{pmatrix}. \tag{11}$$

It is obvious that all eigenfunctions of problem (1)—(10) are expressed linearly in terms of the columns of the matrix $\varphi(x, \lambda)$. Let us find the asymptotics of the matrix $\varphi(x, \lambda)$ as $x \to \infty$. To this end, note that for real λ and $|\lambda| > m$ the matrix function $\overline{f(x, \lambda)}$ is a solution of equation (1), linearly independent of $f(x, \lambda)$. Therefore

$$\varphi(x, \lambda) = \frac{1}{2i}\sqrt{\frac{\lambda - m}{\lambda + m}} \left[\overline{f(x, \lambda)}\, f_1^*(0, \lambda) - f(x, \lambda)\, \overline{f_1^*(0, \lambda)}\right] I_n, \tag{12}$$

where $f_1(0, \lambda)$ is the matrix composed of the first n rows of the matrix $f(0, \lambda)$. From (12) and (5) it is obvious that the asymptotic behavior of the matrix $\varphi(x, \lambda)$, as $x \to \infty$, is determined by the **scattering matrix**

$$f_1^*(0, \lambda) \left[\overline{f_1^*(0, \lambda)}\right]^{-1} = S(\lambda). \tag{13}$$

Under condition (3), problem (1)—(10) has a finite number of discrete eigenvalues $\lambda_1, \dots, \lambda_k$ in $(-m, m)$*, which coincide with the roots of the determinant of the matrix $f_1^*(0, \lambda)$, and the multiplicity of the eigenvalue λ_j coincides with the rank of the matrix $f_1(0, \lambda_j)$. One can prove that there exist positive definite normalizing matrices $M_1, \dots, M_k$ of order n such that

$$\frac{1}{\pi}\int_{|\lambda|>m} \varphi(x, \lambda) I_n \{f_1(0, \lambda) f_1^*(0, \lambda)\}^{-1} I_n \varphi^*(t, \lambda) \sqrt{\frac{\lambda + m}{\lambda - m}}\, d\lambda +$$

$$+ \sum_{j=1}^{k} f(x, \lambda_j) M_j^2 f^*(t, \lambda_j) = \delta(t - y) E_{2n}. \tag{14}$$

Here the rank of the matrix M_j coincides with the multiplicity of λ_j.

* We assume that $\lambda = \pm m$ is not a virtual level.

The collection of quantities $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$ is called the **scattering data** of problem (1)–(10). The inverse problem of scattering theory for a Dirac system of order $2n$ is formulated as follows. Is it possible, knowing the scattering data of a problem of type (1)–(10), to reconstruct equation (1), i.e., to find $Q(x)$? If so, what properties must the collection of quantities $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$ possess in order that it be the scattering data of a problem of type (1)–(10)? In what follows an answer to these questions is given.

4. The main equation and the solution of the inverse problem of scattering theory. Multiply both sides of equality (12) by

$$-\frac{1}{2i\pi} I_n \, [f_1^*(0,\lambda)]^{-1} [f_0(x,\lambda) - f_0(x,\lambda)S(\lambda)]^*$$

and integrate with respect to λ over the intervals $(-\infty, -m)$ and (m, ∞). Then, for $t > x$, from Parseval' s equality (14), transformation (5), and the equality $S^*(\lambda) = S^{-1}(\lambda)$, we obtain

$$F(x+t) + K(x,t) + \int_x^\infty K(x,\xi) F(\xi + t) \, d\xi = 0, \tag{15}$$

where

$$F(x+t) = F_S(x+t) + \sum_{j=1}^{k} f_0(x,\lambda_j) M_j^2 f_0^*(t,\lambda_j), \tag{16}$$

$$\begin{aligned} F_S(x+t) = {} & \frac{1}{4\pi} \int_{|\lambda|>m} f_0(x,\lambda)[E_n - S(\lambda)] f_0^*(t,\lambda) \sqrt{\frac{\lambda - m}{\lambda + m}} \, d\lambda \\ & + \frac{1}{4\pi} \int_{|\lambda|>m} \overline{f_0(x,\lambda)} [E_n - S^*(\lambda)] f_0^*(t,\lambda) \sqrt{\frac{\lambda - m}{\lambda + m}} \, d\lambda. \end{aligned} \tag{17}$$

Equation (15) will be called the **main equation of the inverse problem of scattering theory**. This equation makes it possible to solve the inverse problem formally. Indeed, the self-adjoint matrix function $F(x+t)$ is found solely from the scattering data $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$, and, moreover, equation (15) has a unique matrix solution $K(x,t)$ for each fixed $x \in [0, \infty)$. Using the solution $K(x,t)$, by formula (5) we can find the matrix function $f(x,\lambda)$, which is a solution of an equation of type (1) with

$$Q(x) = B_{nK}(x,x) - K(x,x) B_n.$$

However, the reasoning carried out above is conditional in nature, since we assume in advance that the collection of quantities $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$

represents the scattering data of a problem of type (1)–(10). Therefore it is of interest to find conditions such that the collection of quantities $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$ represents the scattering data of a problem of type (1)–(10). These conditions are given in the following theorem.

Theorem 2. Let a collection of quantities $S(\lambda); \lambda_1, \dots, \lambda_k; M_1, \dots, M_k$ be given, where $S(\lambda)$ is a matrix function of order n, defined on $(-\infty, -m]$ and $[m, \infty)$; $\lambda_1, \dots, \lambda_k$ are real numbers from the interval $(-m, m)$; $M_1, \dots, M_k$ are positive definite matrices of order n. In order that this collection of quantities represent the scattering data of a problem of type (1)–(10) with a matrix function $Q(x)$ of order $2n$ satisfying condition (3), it is necessary and sufficient that the following conditions hold:

1°. $S^*(\lambda) = S^{-1}(\lambda) = \overline{S}(\lambda)$.

2°. Each element F_S^{ij} of the matrix function $F_S(x)$, defined by formula (20), belongs to $L_2(-\infty, \infty)$, and for positive x

$$\|F_S^{ij}(x)\| \leq C/(1+x)^{2+\varepsilon},$$

where

$$p = \begin{cases} 2, & \text{if } i, j \leq n \text{ or } i, j \geq n+1, \\ 1, & \text{if } i \leq n,\ j \geq n+1. \end{cases}$$

3°. For each fixed $x \geq 0$, the homogeneous equation

$$\psi(t) + \int_x^\infty \psi(\xi) F(\xi + t)\, d\xi = 0 \tag{18}$$

has only the zero vector solution $\psi(t)$ with components in $L_2(x, \infty)$. Here $\psi(t)$ is a row vector-function with $2n$ components; $F(x+t)$ is determined through the quantities $S(\lambda)$; $\lambda_1, \dots, \lambda_k$; $M_1, \dots, M_k$ by formulas (16) and (17).

4°. The number of linearly independent vector solutions with components in $L_2(0, \infty)$ of the homogeneous equation

$$\psi(t) + \int_0^\infty \psi(\xi) F_S(\xi + t)\, d\xi = 0 \tag{19}$$

coincides with the sum of the ranks of the matrices $M_1, \dots, M_k$. Here $F_S(x+t)$ is determined through $S(\lambda)$ by formula (17).

5°. The homogeneous equation

$$-\psi(t)+\int_{-\infty}^{0}\psi(\xi)F_S(t+\xi)\,d\xi=0 \qquad (-\infty<t\le 0) \tag{20}$$

has only the zero vector solution with components in $L_2(-\infty,0)$.

5. In conclusion we note that we have also succeeded in solving the inverse scattering problem in the case when the matrix function $Q(x)$ has singularities of the type

$$\begin{pmatrix} 0 & J \\ J^* & 0 \end{pmatrix}\frac{1}{x}$$

at zero and at infinity. Here J is a matrix of order n, all of whose elements are equal to zero, except for the elements lying on the second diagonal, which are equal to integers.

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Note: Figure translations are in progress. See original paper for figures.