

**APPROXIMATION OF
CONTINUOUS
FUNCTIONS BY
HOLOMORPHIC
FUNCTIONS ON
JORDAN ARCS IN
 \mathbb{C}^n**

MATHEMATICS

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.59341>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.55

MATHEMATICS

E. M. CHIRKA

APPROXIMATION OF CONTINUOUS FUNCTIONS BY HOLOMORPHIC FUNCTIONS ON JORDAN ARCS IN C^n

(Presented by Academician M. A. Lavrent'ev on 14 VI 1965)

In this paper we study the possibility of uniform approximation of continuous functions on simple Jordan arcs in the space C^n of complex variables $z = (z_1, \dots, z_n)$ by functions holomorphic on these arcs. By $C(X)$ we denote the algebra of all continuous complex-valued functions on a compact set X , and by $A(X)$, $X \subset C^n$, the algebra of all uniform limits of functions holomorphic on X .

1. **Theorem 1.** *Let γ be a simple Jordan arc in C^2 such that its projection $z_1(\gamma)$ is a nowhere dense subset of C^1 . Then $C(\gamma) = A(\gamma)$.*

Proof. 1) We first show that for every polynomial $p(z_1)$ and every positive integer m , the function

$$\sqrt[m]{p(z_1)}$$

belongs to $A(\gamma)$.

For this it is enough to prove that

$$\sqrt[m]{z_1 - a} \in A(\gamma)$$

for any point $a \in C^1$. Let $a \in z_1(\gamma)$, and let a_ν , $\nu = 1, 2$, be such that $a_\nu \in C^1 \setminus z_1(\gamma)$ and $a_\nu \rightarrow a$ as $\nu \rightarrow \infty$. For each ν , in view of the connectedness of γ , one can construct a function holomorphic on γ ,

$$\sqrt[m]{z_1 - a_\nu}$$

(see (1)). Let $\gamma_a = \{z \in \gamma : z_1 = a\}$, and let E_δ be the subset of γ consisting of a finite number of connected components, containing the set $\{z \in \gamma : |z_1 - a| \geq \delta > 0\}$ and not intersecting γ_a . Then from the sequence of functions $\sqrt[m]{z_1 - a_\nu}$, $\nu = 1, 2, \dots$, one can choose a subsequence converging uniformly on E_δ . Taking $\delta = 1/n$ and letting $n \rightarrow \infty$, by a diagonal process we extract a subsequence

$$\sqrt[m]{z_1 - a_{\nu_k}}, \quad k = 1, 2, \dots,$$

which converges uniformly on any compact subsets of $\gamma \setminus \gamma_a$.

Let $\varepsilon > 0$ be given. Choose a neighborhood U of the set γ_a in C^n such that $|z_1 - a_\nu| < \varepsilon^m$ if $z \in U$ and $a_\nu \in z_1(U)$. Choose k_1 so that $a_{\nu_k} \in z_1(U)$, $k > k_1$, and $\delta > 0$ so that $\gamma \setminus U \subset E_\delta$. In view of the uniform convergence of the sequence $\{\sqrt[m]{z_1 - a_{\nu_k}}\}$ on E_δ , there is k_2 such that

$$\left| \sqrt[m]{z_1 - a_{\nu_{k+1}}} - \sqrt[m]{z_1 - a_{\nu_k}} \right| < \varepsilon$$

for $k > k_2$. Then for $k > \max(k_1, k_2)$ we obtain, uniformly on γ ,

$$\left| \sqrt[m]{z_1 - a_{\nu_{k+1}}} - \sqrt[m]{z_1 - a_{\nu_k}} \right| < 2\varepsilon,$$

whence it follows that

$$\sqrt[m]{z_1 - a} \in A(\gamma),$$

and therefore

$$\sqrt[m]{p(z_1)} \in A(\gamma).$$

Consider the algebra generated by functions of the form

$$\sqrt[m]{p(z_1)},$$

where p is an arbitrary polynomial and m is any natural number. In a completely analogous way it is proved that roots of any degree of functions of this algebra belong to $A(\gamma)$. From the roots obtained we construct a new algebra, etc. The union of all these algebras forms a subalgebra $A_1 \subset A(\gamma)$, which, together with each f , also contains

$$\sqrt[m]{f}$$

for any natural m . By a diagonal process, as was done above, this property can be extended to the uniform closure \bar{A}_1 of the algebra A_1 on γ .

- 2) Let V and E be connected compact subsets of γ such that $|z_1 - a| < \alpha$ for $z_1 \in z_1(V)$ and $|z_1 - a| > \alpha$ for $z_1 \in z_1(E)$. Put

$$f_m = \alpha \sqrt[m]{\left(\frac{z_1 - a}{\alpha}\right)^m + 1};$$

we have

$$\lim_{m \rightarrow \infty} |f_m| = \max(|z_1 - a|, \alpha)$$

and

$$\lim_{m \rightarrow \infty} f_m(z) = \alpha$$

for $z \in V$, in view of the connectedness of V . Let

$$f_{mN} = \sqrt[N]{f_m - \alpha};$$

then

$$\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} f_{mN}(z) = 0,$$

if $z \in V$. Since the variation of the argument $\Delta_E \text{Arg}(f_m - \alpha)$ on the connected compact set E does not exceed $\Delta_E \text{Arg}(z_1 - a) + 2\pi$, choosing the necessary branches, we may assume that

$$\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} f_{mN}(z) = 1,$$

if $z \in E$. Therefore, for any $\delta' > 0$ one can choose m and N so that $|f_{mN}| < 1 + \delta'$, $|f_{mN}(z) - 1| < \delta'$, $z \in E$, and $|f_{mN}(z)| < \delta'$, $z \in V$.

Let now U_a be a neighborhood of γ_a in γ , consisting of a finite number of connected components $U_{a\nu}$, $\nu = 1, \dots, n$, containing the set $\{z \in \gamma : |z_1 - a| \leq \alpha, \alpha > 0\}$ and contained in $\{z \in \gamma : |z_1 - a| < \alpha + \beta, \beta > 0\}$. Let E_a be a subset of γ , consisting of a finite number of connected compact sets $E_{a\mu}$, such that

$$\{z \in \gamma : |z_1 - a| \geq \alpha + 2\beta\} \subset E$$

and

$$E \subset \{z \in \gamma : |z_1 - a| > \alpha + \beta\}.$$

In view of the local connectedness of γ , such sets can be constructed for any $\alpha, \beta > 0$. For each pair $U_{a\nu}, E_{a\mu}$ and any $\delta' > 0$ we can now construct a function $g_{\mu\nu}$ such that $|g_{\mu\nu}| < 1$, $|g_{\mu\nu}(z) - 1| < \delta'$, $z \in E_{a\mu}$, and $|g_{\mu\nu}(z)| < \delta'$, $z \in \overline{U}_{a\nu}$.

Let

$$g_\nu = \prod_{\mu} g_{\mu\nu}$$

and

$$g = 1 - \sqrt[n]{(1 - g_1) \cdots (1 - g_n)}.$$

Then, if $\delta'' > 0$, δ' can be chosen so that $|g| \leq 1$, $|g(z) - 1| < \delta''$, if $z \in E$, and $|g(z)| \leq \delta''$, if $z \in \overline{U}_a$. Let $1 > \delta > 0$, and let φ be a fractional-linear transformation of the unit disk onto itself such that $\varphi(0) = -1 + \delta$ and $\varphi(1) = 1$. Choose δ'' so that $|\varphi(\lambda) - 1| < \delta^4$, if $|\lambda - 1| < \delta''$, and $|\varphi(\lambda) + 1| < 2\delta^4$, if $|\lambda| < \delta''$, and put

$$h_a = \sqrt[4]{\frac{1}{2}(1 - \varphi \circ g)};$$

according to 1), $h_a \in A(\gamma)$. Moreover:

$$|h_a| \leq 1, \quad |\arg h_a| \leq \pi/8,$$

$$|h_a(z) - 1| < \delta, \quad \text{if } z \in \overline{U}_a,$$

and

$$|h_a(z)| < \delta, \quad \text{if } z \in E_a.$$

Let now $f \in C(\gamma)$ and $\varepsilon > 0$. By the theorem of M. A. Lavrentiev ⁽²⁾, for every $a \in z_1(\gamma)$ there is a polynomial

$$p_a(z) \equiv p_a(z_1)$$

such that $|f(z) - p_a(z)| < \varepsilon$, if $z \in \gamma_a$. Choose α and β so that $|f - p_a| < \varepsilon$ on $\gamma \setminus E_a$. From the covering of γ by the sets U_a select a finite subcover: U_1, \dots, U_k . Let p_1, \dots, p_k be the corresponding polynomials and h_1, \dots, h_k the functions constructed above for δ and for the pairs U_i, E_i ; we assume that $0 < \delta < 1/2$. Put

$$e_i = h_i(h_1 + \dots + h_k)^{-1}.$$

Then

$$e_1 + \dots + e_k \equiv 1$$

on γ , and

$$|\arg e_i| < \pi/4, \quad i = 1, \dots, k.$$

Moreover,

$$|e_i(z)(f(z) - p_i(z))| < \varepsilon, \quad z \in \gamma \setminus E_i,$$

and

$$|e_i(z)(f(z) - p_i(z))| < 2 \max_{\gamma} |f - p_i| \cdot \delta, \quad z \in E_i.$$

Let $z \in \gamma$, and let i_1, \dots, i_s be all indices such that $z \notin E_{i_\nu}$, $\nu = 1, \dots, s$. Then

$$\begin{aligned} \left| f(z) - \sum_{i=1}^k e_i(z)p_i(z) \right| &= \left| \sum_{i=1}^k e_i(z)(f(z) - p_i(z)) \right| \leq \\ &\leq \left| \sum_{\nu=1}^s e_{i_\nu}(z)(f(z) - p_{i_\nu}(z)) \right| + \left| \sum_{\nu=s+1}^k e_{i_\nu}(z)(f(z) - p_{i_\nu}(z)) \right| < \\ &< \varepsilon \sum_{\nu=1}^s |e_{i_\nu}(z)| + \left(2k \max_i \max_{\gamma} |f - p_i| \right) \delta. \end{aligned}$$

Since

$$\sum_{\nu=1}^s |e_{i_\nu}(z)| \leq \sqrt{2},$$

with a proper choice of δ we obtain uniformly on all of γ

$$\left| f - \sum_{i=1}^k e_i p_i \right| < \varepsilon.$$

Since

$$\left(\sum_{i=1}^k e_i p_i \right) \in A(\gamma),$$

the theorem is proved.

2. Theorem 2. *Let γ be a simple Jordan arc in C^2 . Then, for every function j holomorphic on γ , either the set $j(\gamma)$ contains interior points, or $C(\gamma) = A(\gamma)$.*

In the proof of Theorem 1 we used Lavrentiev's theorem on approximation by polynomials on plane compact sets. The proof of Theorem 2 rests on the following generalization of S. N. Mergelyan's theorem ⁽³⁾:

Theorem 3. *Let O be an open subset of C^n and S a one-dimensional analytic set in O . Let a compact set $X \subset S$ not divide the irreducible components of this set. Then: 1) the set X is polynomially convex in C^n ; 2) every function continuous on X and holomorphic at the points of X that are interior relative to S is uniformly approximable on X by polynomials in z_1, \dots, z_n .*

Theorem 3 is proved using results of Rossi ⁽⁴⁾ and Bishop ⁽⁵⁾. It is interesting to compare Theorem 2 with the known examples of Wermer ⁽⁶⁾ and Rudin ⁽⁷⁾ of Jordan arcs in C^3 and C^2 on which not every continuous function is approximable by polynomials. However, in these examples, for every polynomial $p(z)$, the plane set $p(\gamma)$ contains no interior points.

3. **Definition.** Let A be a multiplicative group. We shall say that A is **closed under extraction of (square) roots** if, together with every $f \in A$, A also contains an element \sqrt{f} such that $(\sqrt{f})^2 = f$. Let X be an arbitrary compact set, and let A_0 be a subset of $C(X)$. By the **algebraic extension** of the set A_0 we mean the minimal subalgebra of $C(X)$ that contains A_0 , is closed with respect to uniform convergence on X , and is closed with respect to the operation of extracting square roots.

In the first part of the proof of Theorem 1, essentially the following was proved.

Theorem 4. *Let X be a locally connected compact set, and let A be some family of functions continuous on X , closed with respect to extraction of roots. Then \overline{A} (the uniform closure of A on X) is also closed with respect to this operation.*

In the second part of the proof of Theorem 1 one can get by with roots of powers of the form 2^k , where k is a natural number. In doing so, only the following facts are essentially used: 1) the arc γ is locally connected; 2) there exists a family A_0 of functions from $A(\gamma)$ (namely, the functions z_1), whose algebraic extension lies in $A = A(\gamma)$; 3) on every set γ_a , where all functions from A_0 are constant, $C(\gamma_a) = A|_{\gamma_a}$ ($A|_{\gamma_a}$ denotes the restriction of A to γ_a).

Thus the following holds.

Theorem 5. *Let X be a locally connected compact set; let A and B , $A \subset B$, be algebras of continuous functions on X . Suppose that in A there exists a subset A_0 such that: 1) the algebraic extension of A_0 lies in A ; 2) on every subset $X_a \subset X$ on which all functions from A_0 are constant, $\overline{B|_{X_a}} = \overline{A|_{X_a}}$. Then $\overline{A} = \overline{B}$.*

If A_0 separates the points of X , then condition 2) is automatically satisfied, and therefore the following holds.

Theorem 5'. *Let A be a separating algebra of continuous functions with identity on a locally connected compact set X , closed with respect to extraction of square*

roots. Then $\bar{A} = C(X)$.

In conclusion I express my deep gratitude to A. G. Vitushkin, A. A. Gonchar, and B. V. Shabat for discussion of this work.

Moscow State University
named after M. V. Lomonosov

Received
16 IV 1965

CITED LITERATURE

- ¹ G. Stolzenberg, *Acta Math.*, **109**, 3–4 (1963).
- ² M. A. Lavrentiev, *Actual Sci. et Ind.*, No. 441, Paris, 1936.
- ³ S. N. Mergelyan, DAN, **78**, No. 3 (1951).
- ⁴ H. Rossi, *Trans. Am. Math. Soc.*, **100**, 3 (1961).
- ⁵ E. Bishop, *Trans. Am. Math. Soc.*, **102**, 3 (1962).
- ⁶ J. Wermer, in: *Some Questions of Approximation Theory*, IIL, 1963, p. 120.
- ⁷ W. Rudin, *ibid.*, p. 124.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.