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Abstract

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MATHEMATICS

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ON THE MOMENTS OF THE NUMBER OF CLASSES OF PURELY RADICAL QUADRATIC FORMS OF NEGATIVE DETERMINANT

(Presented by Academician Yu. V. Linnik, 17 VI 1965)

The question of the order of the sum

$$\sum_{D\leq N} h^n(-D), \tag{1}$$

where $h(-m)$ is the number of classes of purely radical quadratic forms of negative determinant, has been studied by many authors. For the case $n = 1$, I. M. Vinogradov as early as 1917 (^{1}) derived an asymptotic formula with the separation of two main terms (for the latest results see (^{2})). The cases $n = 2, 3$ were considered by A. F. Lavrik (^{3}), and the case $n = 4, 5$ by O. Sarnak (^{4}); moreover, for these cases the remainder term in (1) had a power saving in comparison with the main term.

For arbitrary n, in (^{5}) the formula was proved

$$\sum_{D\leq N} h^n(-D) = \frac{2^{n+1}}{\pi^n(n+2)}\, r(n)\, N^{(n+2)/2} + O\left(N^{(n+2)/2}e^{-\ln^{1/2-\varepsilon} N}\right),$$

where

$$r(n) = \sum_{\substack{k=1\\ k=1\,(\mathrm{mod}\,2)}}^{\infty} \frac{\varphi(k)\tau_n(k^2)}{k^3}.$$

A fundamental role in its proof was played by A. Rényi' s theorem (^{6}) on the displacement of the zeros of Dirichlet L-series for almost all moduli, obtained with the aid of Yu. V. Linnik' s "large sieve" method.

In Yu. V. Linnik' s work (^{7}), ideas were put forward that proved useful for the theory of Dirichlet L-series for almost all moduli. In particular, A. I. Vinogradov in (^{8}), combining these ideas with his own ingenious considerations, proved the validity of an important mean density hypothesis for additive problems. The following inequality played the main role in these works:

for $n \geq 2$

$$\sum_{M<D\leq 2M} \sum_{\chi\neq\chi_0} \left|\sum_{m\leq Z} \chi_D(m)\right|^{2n} \ll M^2 Z^n \exp(\ln M)^\varepsilon, \tag{2}$$

where $M^{1/n} \leq Z \leq M^{1/(n-1)}$, which we shall henceforth call the A. I. Vinogradov–Yu. V. Linnik inequality.

In the present note we show that the problem of the moments of the number of classes of purely radical quadratic forms of negative determinant, even with a power saving in the remainder term, is a simple consequence of the A. I. Vinogradov–Yu. V. Linnik inequality.

Theorem. For any fixed n, the following asymptotic equality holds:

$$\sum_{D<N} h^n(-D) = \frac{2^{n+1}}{\pi^n(n+2)} r(n) N^{(n+2)/2} + O\Big(N^{(n+2)/2(1-1/5n^2)}\Big).$$

Proof. Gauss' s formula

$$h(-D) = \frac{2}{\pi}\sqrt{D}\, L(1, D)$$

allows us to confine ourselves to studying the sum of powers of values of L-series at a single point.

The following formula (see (3)) reduces this question to estimates of L-series in the critical strip:

$$\sum_{D<N} L^n(1,\chi_D) = r(n)N - \frac{1}{2\pi i} \sum_{D<N} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s-1) L^n(s,\chi_D) P^{s-1}\, ds + O(P^{1/2+\varepsilon}) + O(NP^{-1/2+\varepsilon}), \tag{3}$$

where $1/2 < \gamma < 1$, and $\varepsilon > 0$ is arbitrarily small.

But the inequality of A. I. Vinogradov—Yu. V. Linnik makes it possible to estimate the L-series well in the critical strip for almost all moduli. Indeed, it follows from (2) that for all $M \leq D \leq 2M$, with the exception of no more than $M^{1-\alpha}$ moduli D, for all nonprincipal characters $\chi_D(\mathrm{mod} D)$ the inequality

$$S(Z) = \sum_{m<Z} \chi_D(m) \ll Z^{1/2} M^{1/2n+\alpha}, \tag{4}$$

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holds, where $M^{1/n} \leq Z \leq M^{1/(n-1)}$, $n \geq 2$, and the constant implied by the symbol $\ll$ depends on n.

For nonexceptional moduli, using Abel' s transformation and estimate (4), we obtain

$$L(s, \chi_D) \ll (|t| + 2) \left(M^{(1-\sigma)/n} + M^{\alpha - 1/2n(n-1) + 1 - \sigma} \right). \tag{5}$$

We now split the interval of summation in the right-hand side of (3) into subintervals of the form $(2^N/(k+1), 2^N/k)$.

For exceptional D we shall use the crude estimate

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s-1) L^k(s, \chi_D) P^{s-1}\, ds = (\ln^n D).$$

We obtain

$$\sum_{i < \ln N} \sideset{}{'}\sum_{N/2^{k+1} < D < N/2^k} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s-1) L^n(s, \chi_D) P^{s-1}\, ds \ll N^{1-\alpha} (\ln N)^{n+1}, \tag{6}$$

where the prime denotes summation over exceptional moduli.

For nonexceptional D, using (5), we obtain

$$\sum_{D<N} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s-1) L^k(s, \chi_D) P^{s-1}\, ds \ll$$

$$\ll \left(N^{2-\gamma} P^{\gamma - 1} + N^{(\alpha - 1/2n(n-1) + 1 - \gamma)n + 1} P^{\gamma - 1} \right) \ln N, \tag{7}$$

where the constant implied by the symbol $\ll$ depends on n.

Putting in (3), (6), and (7) $\alpha = 1/4n^2 = 1 - \gamma,\ P = N^{2 - 1/2n^2}$, we obtain

$$\sum_{D \leq N} L^n(1, \chi_D) = r(n) N + O(N^{1 - 1/5n^2}). \tag{8}$$

Now the theorem is obvious.

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CITED LITERATURE

[1] I. M. Vinogradov, *Communications of the Kharkov Mathematical Society*, **16**, 10 (1917).
[2] I. M. Vinogradov, *Izv. AN SSSR, Ser. Mat.*, **27**, 7 (1963).
[3] A. F. Lavrik, *Izv. AN UzSSR, Ser. Phys.-Math. Sci.*, No. 1, 84 (1959).
[4] O. Sands, *Izv. AN UzSSR, Ser. Phys.-Math. Sci.*, No. 4, 88 (1964).
[5] M. B. Barban, *Izv. AN SSSR, Ser. Mat.*, **26**, 4, 573 (1962).
[6] A. Rényi, *Izv. AN SSSR, Ser. Mat.*, **12**, 1, 57 (1948).
[7] Yu. V. Linnik, *Izv. AN SSSR, Ser. Mat.*, **24**, 629 (1960).
[8] A. I. Vinogradov, *Izv. AN SSSR, Ser. Mat.*, **29**, 4 (1965).

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