DUALITY IN MEASURE THEORY IN LINEAR SPACES

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MATHEMATICS

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DUALITY IN MEASURE THEORY IN LINEAR SPACES

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1°. Let E be a real locally convex weakly separable linear topological space; E' the space of continuous linear functionals on E; μ a complete normalized completely additive nonnegative measure defined on the σ-algebra of weakly Borel subsets of E. Suppose that (E, μ) is a Lebesgue space * ([1]). Under these conditions we shall call the pair (E, μ) a **linear space with measure**. Two linear spaces with measure are called **linearly isomorphic** if there exists between them an isomorphism in the sense of measure theory which is linear mod 0 **.

Introduce the space $S_\mu(E') = S$ of classes, coinciding mod 0, of measurable finite almost everywhere real-valued functions on (E, μ), and endow it with the topology of convergence in measure. S is a linear topological ring. E' can be regarded as a linear manifold in S. It is easy to establish that E' is generating in S. (A subset $R \subset S$ is called **generating** if the minimal closed subring spanned by it coincides with all of S.) Denote the closure of E' in S by $\mathcal{L}_\mu = \mathcal{L}$. In $\mathcal{L}$ we introduce the topology from S. It can be shown that if (E_1, μ_1) and (E_2, μ_2) are linearly isomorphic, then there exists a ring (multiplicative, linear, continuous) isomorphism between S_1 and S_2 carrying $\mathcal{L}_1$ onto $\mathcal{L}_2$. Every statement concerning the pair (E, μ) can be expressed in terms of the pair $(S, \mathcal{L})$, and conversely; the study of (E, μ) up to linear isomorphism is equivalent to the study of $(S, \mathcal{L})$ up to a ring isomorphism leaving $\mathcal{L}$ fixed. In measure theory in linear spaces $\mathcal{L}$ plays the same role as the conjugate space in the theory of locally convex spaces: $\mathcal{L}$ is the space (of classes) of all measurable linear functionals on (E, μ). Instead of the assumptions of local convexity of E, one could, without harm to the whole theory, assume only the existence of a sufficient number of linear measurable functionals. Note also that the pair $(S, \mathcal{L})$ may be perceived independently of (E, μ), namely: $S = S_m(X)$, where (X, m) is an abstract Lebesgue space, $\mathcal{L}$ is an arbitrary closed generating subspace of S; specifying $\mathcal{L}$ is exactly equivalent to introducing a linear structure in (X, m) with respect to which $\mathcal{L}$ is the set of all measurable linear functionals (see ([2]), where the case $S = \mathcal{L}$ is considered). The purpose of the present note is to show the convenience of the duality of (E, μ) and $(S, \mathcal{L})$, emphasizing the transition from the geometry of (E, μ) to $(S, \mathcal{L})$. Our approach makes it possible, along with new results, to establish known facts; here properties of invariant and quasi-invariant measures in linear spaces are considered.

2°. Denote by $\mathcal{L}'$ the space conjugate to $\mathcal{L}$; since $\mathcal{L}$, generally speaking, is not locally convex, it may happen that $\mathcal{L}' = \{0\}$. Introduce into consideration $H_\mu = H = \mathcal{L} \cap L^2_\mu$, where L^2_μ is the Hilbert space of real-valued functions on (E, μ) that are square-integrable. In H we introduce

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* I.e. isomorphic, in the sense of measure theory, to the segment $[0, 1]$ with Lebesgue measure with no more than a countable number of atoms.

** Linearity here and below is a synonym for additivity and homogeneity; continuity is not assumed here.

the topology L^2_μ. H may also be trivial, $H = \{0\}$, since not for every measure in a linear space do there exist square-integrable linear functionals. We note that $H \subset \mathcal{L}$. We shall call a linear space with measure $(\mathcal{H}, \mu)$ **Hilbertian** if $\mathcal{H}$ is a separable Hilbert space.

Theorem 1. *Every linear space with measure is linearly isomorphic to a Hilbertian space with measure.*

We outline the proof. Let (E, ν) be the original space with measure. Suppose first that H is generating. Introduce the characteristic functional

$$\chi_\nu(f) = \int e^{if(x)}\, d\nu.$$

Since it is continuous on H, it thereby defines a weak distribution μ_0 in H', which extends to a measure μ in some (for example, nuclear ([4],[5])) extension of the space H'—$\mathcal{H}$. $\mathcal{H}$ is a separable Hilbert space. Since $H_\mu = H_\nu = H$ and H is generating, the identical isomorphism from H can be extended to a ring isomorphism S_ν and S_μ, to which there corresponds a linear isomorphism $P : (E, \nu) \to (\mathcal{H}, \mu)$. If H is not generating, then one may pass from the measure ν to a measure $\tilde{\nu}$, mutually absolutely continuous with it (briefly, equivalent), such that $H_{\tilde{\nu}}$ is already generating in $S_{\tilde{\nu}} = S_\nu$; the linear isomorphism $P(E, \tilde{\nu})$ and $(\mathcal{H}, \mu)$ constructed as above is measurable with respect to ν. Denoting $P\nu = \mu$, we obtain that (E, ν) and $(\mathcal{H}, \mu)$ are linearly isomorphic. (In what follows we assume H to be generating, and hence $\mathcal{L}' \subset H'$.)

Remark A. Let us point out the embeddings $\mathcal{L}' \subset H' \subset \mathcal{H}$ and $\mathcal{H}' \subset H \subset \mathcal{L}$. Although the choice of the extension $\mathcal{H}$ is not unique, the embeddings $\mathcal{L}' \subset \mathcal{H}$ and $\mathcal{H}' \subset \mathcal{L}$ have a quite definite character: there exists a Hilbert space $\hat{H}$ such that

$$\mathcal{H}' \subset \hat{H} \subset \mathcal{L} \quad (\mathcal{L}' \subset \hat{H}' \subset \mathcal{H})$$

and the embedding $\mathcal{H}' \subset \hat{H}$ $(\hat{H}' \subset \mathcal{H})$ is effected by a Hilbert-Schmidt operator (i.e. is an S-embedding). Indeed, as $\hat{H}$ one may take the completion of $\mathcal{H}'$ with respect to the norm

$$\|h\|^2 = \int_{(x,x)_H < K} h(x)^2\, d\mu, \qquad \mu\{x : (x,x)_H \le K\} > 0.$$

This argument is, in essence, the proof of necessity in the Minlos–Sazonov theorem ([4],[5]). We note that, generally speaking, $H \neq \hat{H}$, and the embedding $H' \subset \mathcal{H}$ ($\mathcal{H}' \subset H$) does not have so definite a form.

Remark B. Since $\mathcal{L}' \subset \mathcal{H}$, one may ask about the measurability and measure of $\mathcal{L}'$ in $\mathcal{H}$. We shall show that $\mu\mathcal{L}' > 0$ if and only if $\mathcal{L}$ (and $\mathcal{H}$) is finite-dimensional. Let $\mathcal{L}'$ be measurable and $\mu\mathcal{L}' > 0$; then $\mu\hat{H}' > 0$, and moreover (restricting the measure to $\hat{H}$) one may assume that $\mu\hat{H} = 1$; hence $\mathcal{H} = \hat{H}$, but the identity operator is a Hilbert-Schmidt operator only in the finite-dimensional case.

Theorem 1 allows one to restrict attention to Hilbertian spaces with measure.

3°. We shall call an **automorphism** of the space $(\mathcal{H}, \mu)$ a one-to-one mod 0 mapping $T(\mathcal{H}, \mu)$ onto itself such that the image and the preimage of a measurable set are measurable and the measures $T\mu$ and μ are equivalent. As usual, associate with T a ring automorphism $U_T : S_\mu \to S_\mu$,

$$(U_T f)(x) = f(Tx).$$

If $T\mu = \mu$, then U_T is isometric in S_μ, $U^2_{TL_\mu} = L^2_\mu$, and is orthogonal in L^2_μ. We shall be interested only in affine automorphisms, i.e. those which preserve the affine structure in $(\mathcal{H}, \mu)$. For them

$$U_T \hat{\mathcal{L}}_\mu = \hat{\mathcal{L}}_\mu,$$

where $\hat{\mathcal{L}}_\mu = \mathcal{L}_\mu + \{c1\}$; denote the restriction $U_T|_{\hat{\mathcal{L}}} = u_T$; we have $u_T 1 = 1$, and for $f \in \mathcal{L}_\mu$

$$u_T f = Vf + \gamma(f)1,$$

where V and γ are continuous (by virtue of the continuity of U_T and u_T) linear operator $V : \mathcal{L}_\mu \to \mathcal{L}_\mu$ and functional $\gamma : \mathcal{L}_\mu \to R^1$. In this subsection we shall examine the case when $V = I$, I being the identity operator in $\mathcal{L}_\mu$. We shall call an automorphism $T = T_\gamma$ a **measurable shift** if

$$u_T f = f + \gamma(f)1.$$

Since $\gamma \in \mathcal{L}'_\mu$ and $\mathcal{L}'_\mu \subset \mathcal{H}$, T_γ is a shift in $\mathcal{H}$ by the element γ in the usual sense of the word. Denote

$$G_\mu = \{\gamma :\ T_\gamma \text{ is a measurable shift}\}.$$

We shall call G_μ the **set of quasi-invariance** of μ.

Theorem 2. $G_\mu \subset \mathcal{L}'_\mu$.

From this obvious fact, obtained above, the known results on the properties of G_μ immediately follow.

Corollary A. *The set of quasi-invariance of any measure μ in $\mathcal{H}$ is contained in some Hilbert space S-embedded in $\mathcal{H}$ ([6],[7]).*

This corollary follows from Theorem 2 and Remark A in § 2.

Corollary B. *For every nondegenerate measure in an infinite-dimensional linear space, $\mu G_\mu = 0$ ([8],[9]).*

This corollary follows from Remark B in § 2 and Theorems 1 and 2.

Let us study measurable shifts in more detail. Let $\gamma \in G_\mu$. Since H is generating, H' is nontrivial and $\gamma \in \mathcal{L}'_\mu \subset H'$; hence γ may be identified with a certain function from H; we shall retain for it the notation γ. What are the properties of the function γ, if $\lambda\gamma \in G_\mu$, i.e. if the shift $T_{\lambda\gamma}$ is measurable for all $\lambda \in R^1$? Let ξ_f be the partition into inverse images of points with respect to the measurable function f, $f : \mathcal{H} \to R^1$. Denote $\xi = \xi_\gamma$;

$$\gamma^\perp = \{f : \ f \in H, \ (\gamma, f)_H = 0\}; \qquad \eta = \prod_{f \in \gamma^\perp} \xi_f$$

($\prod$ denotes the product of partitions); $\mathcal{H}/\xi$, $\mathcal{H}/\eta$ are the quotient spaces with respect to the partitions ξ and η; μ_ξ and μ_η are the corresponding quotient measures. $(\mathcal{H}/\xi, \mu_\xi)$ is naturally identified with (R^1, μ_γ), where μ_γ is the measure generated by the mapping $\gamma : \mathcal{H} \to R^1$. As follows from the formula $u_{T_{\lambda\gamma}} f = f + \lambda(\gamma, f)_H 1$, the partition η is fixed under all $T_{\lambda\gamma}$, $\lambda \in R^1$, while the quotient automorphism $T_{\lambda\gamma}/\xi$ acts on (R^1, μ_γ) as a shift by the number $\lambda(\gamma, \gamma) = \lambda\|\gamma\|_H^2$; moreover the measure μ_γ, under all shifts, goes over into an equivalent one and, consequently, is equivalent to Lebesgue measure on the line. If one takes into account that ξ and η are mutually complementary and that, by virtue of the transitivity of the action of shifts, the conditional measures of the partition ξ have one and the same metric type (1), then we obtain that μ is equivalent to the direct product of the measures μ_γ and μ_η.

Theorem 3. *In order that $T_{\lambda\gamma}$ be a measurable shift for all $\lambda \in R^1$, it is necessary and sufficient that: 1) the measure μ_γ be equivalent to Lebesgue measure on the line; 2) μ be equivalent to $\mu_\gamma \times \mu_\eta$.*

Necessity was proved above; sufficiency is obvious.

Remark A. An analogous way one can formulate the condition of measurability of the shift T_γ in terms of γ. In this case μ_γ need not be equivalent to Lebesgue measure.

Remark B. Condition 2) can be replaced by an equivalent ring condition: S decomposes into the direct product of the rings spanned by γ and $\gamma^\perp$. The condition of measurability of T_γ in these terms means that $u_T : \hat{\mathcal{L}} \to \hat{\mathcal{L}}$ extends to an automorphism of S. Thus the theorem is given a ring-theoretic character, which makes it possible to avoid the space with measure.

We shall call a measure μ **Hilbert** if $\mathcal{L}_\mu = H$. For Hilbert measures in linear spaces, every measurable linear functional is square-integrable. From Banach' s theorem on the open mapping it follows that in this case on $\mathcal{L}_\mu$ the topologies induced from S and L^2_μ coincide. Therefore $\mathcal{L}'_\mu = H' = H = \mathcal{L}_\mu$. The measure is called **quasi-invariant** if it is Hilbert and $G_\mu = \mathcal{L}'(= H)$. Quasi-invariant measures have the greatest possible supply of measurable shifts. It follows from Theorem 3 that, for a quasi-invariant measure, conditions 1 and 2 must be satisfied for all $\gamma \in \mathcal{L}$. However, this by no means signifies that μ is equivalent to a direct product of measures in R^1. Examples are given below.

1) A measure μ is called **Gaussian** if every functional from $\mathcal{L}$ has a Gaussian distribution. Obviously, $\mathcal{L} = H$. Let $\gamma \in H$; then from the orthogonality of γ and $\gamma^\perp$ there follows, as is known, also independence, and condition 2) of Theorem 3 is satisfied; thus Gaussian measures are quasi-invariant.

2) A measure μ is called **spherically invariant** if the restriction of the characteristic functional χ_μ to H has the form $\chi_\mu(f) =$

$= l(\|f\|)$. By Schoenberg' s theorem,

$$l(t) = \int_0^\infty e^{-ct^2}\, d\rho(c) \tag{11}$$

and a spherically invariant measure is an "integral" over Gaussian measures. Hence their quasi-invariance follows.

3) Let f_i be a basis in H, and let all f_i be independent and identically distributed. If there exists a distribution density $p(\lambda) > 0$ almost everywhere and $p \in W^1_2(R^1)$, then μ is quasi-invariant. In the case under consideration, the latter condition is apparently close to necessary.

One can give more complicated examples showing that the stock of quasi-invariant measures (even if one restricts oneself to irreducible ones, i.e., those for which the aggregate of shifts acts ergodically) is very extensive. It is also easy to propose examples of non-quasi-invariant measures for which G_μ is sufficiently large (for example, generating in S_μ). The basic apparatus in these considerations is a generalization of theorems of the Kakutani type ([12]) and of Andersen and Jessen ([13]).

4°. Let us pass to linear automorphisms. In this case (see item 3) $u_T\mathcal{L} = \mathcal{L}$; we restrict ourselves to the case when $u_T H = H$. If T preserves the measure, then u_T is orthogonal in H. This condition, necessary for preservation of the measure, is also sufficient in the case of spherically invariant measures (including Gaussian ones). Moreover, the following holds.

Theorem 4. *In order that a measure be spherically invariant, it is necessary and sufficient that: a) for every orthogonal operator u in H there exist a measure-preserving automorphism T, with $u = u_T$, or a′) every orthogonal operator in H extends to an automorphism of the ring S.*

Sufficiency was proved in ([10]). Necessity follows from the expression for the characteristic functional (item 3).

Let now T be an arbitrary linear automorphism of $(\mathfrak{H}, \mu)$. What can u_T be? (Or which $u : H \to H$ extend to an automorphism of S?) If μ is Gaussian, then Feldman' s theorem, repeatedly foreshadowed, says that the necessary and sufficient condition in this case is $uu^* = I + \Gamma$, where Γ is a Hilbert-Schmidt operator whose spectrum lies to the right of -1 ([14]). In the general case the situation is less definite.

Theorem 5. *Let A be an arbitrary positive definite operator in a real Hilbert space. There exist a measure μ in $\mathfrak{H}$ and a linear automorphism T of it such that $u_{T_u}T^*$ is spectrally isomorphic to A. If A is bounded, then μ and T can be chosen so that $d(T\mu)/d\mu \leq K < \infty$ almost everywhere.*

Thus, the stock of linear automorphisms is very different for different classes of measures. Let us note that for spherically invariant measures (item 3), for which in the formula for the characteristic functional $\rho(dc) = q(c)\,dc$, $c > 0$, $q(c) > 0$ almost everywhere, the homotheties λI are automorphisms (in general, ρ is a conditional measure on the rays issuing from the origin; for Gaussian measures, as V. N. Sudakov observed, these conditional measures are unit point masses). It would be interesting to describe all linear automorphisms of a direct product of measures.

The duality considered in this note is useful for studying the problem of extension of measures in linear spaces, etc.

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Note: Figure translations are in progress. See original paper for figures.