BOUNDARY-VALUE PROBLEMS FOR ABSTRACT ELLIPTIC EQUATIONS

MATHEMATICS

1966

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Abstract

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UDC 517.948.35

MATHEMATICS

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BOUNDARY-VALUE PROBLEMS FOR ABSTRACT ELLIPTIC EQUATIONS

(Presented by Academician I. G. Petrovskii, January 10, 1966)

On $(0, T)$ consider the equation

$$\ddot{x}(t) - B(t)x(t) = h(t). \tag{1}$$

Here $x(t)$ is the unknown, and $h(t)$ is a known function, defined on $[0, T]$ with values in a Banach space E. Differentiation is understood in the strong sense. For each $t \in [0, T]$, the operator $B(t)$ is a linear operator in E with domain of definition $D(B(t))$. By a solution of equation (1) we shall mean a function $x(t)$ such that: 1) $x(t)$ is strongly continuous on $[0, T]$ and twice strongly continuously differentiable on $(0, T)$; 2) $x(t) \in D(B(t))$; 3) $x(t)$ turns (1) into an identity for $t \in (0, T)$. For equation (1) we consider the boundary-value problem: find a solution of equation (1) satisfying the boundary conditions

$$\gamma_i(x) \equiv \gamma_{i1}x(0) + \gamma_{i2}x(T) + \gamma_{i3}\dot{x}(0) + \gamma_{i4}\dot{x}(T) = a_i. \tag{2}$$

Here $i = 1, 2$; $a_i \in E$; γ_{ij} are numbers. It is assumed here that $x(t)$ has one-sided derivatives at $t = 0$ or at $t = T$, if these derivatives enter into (2).

The most important are:

A_1. The first boundary-value problem: $x(0) = a_1,\ x(T) = a_2$.

A_2. The second boundary-value problem: $\dot{x}(0) = a_1,\ \dot{x}(T) = a_2$.

A_3. The problem with periodic conditions $x(0) = x(T),\ \dot{x}(0) = \dot{x}(T)$.

Let us formulate precise conditions on the parameters of equation (1). Suppose the following conditions are satisfied:

I. For each $t \in [0, T]$, $B(t)$ is a closed linear operator in E with domain of definition $D(B(t))$ dense in E, and for all $\lambda \geq 0$ the inequality

$$\|[B(t) + \lambda I]^{-1}\| \leq M/\lambda + 1$$

holds.

II. Define $B^{1/2}(t)$ (see (^{1}))—a closed linear operator in E with domain of definition $D(B^{1/2}(t))$ dense in E. Let $D(B^{1/2}(t)) \equiv D$ not depend on t. Assume also that for $t, \tau \in [0, T]$ the operator $B^{1/2}(t)B^{-1/2}(\tau)$ is continuously differentiable with respect to t in the uniform operator topology and $\dot{B}^{1/2}(t)B^{-1/2}(\tau)$ satisfies a Hölder condition with exponent $\gamma \in (0, 1]$, i.e., for $t, s, \tau \in [0, T]$,

$$\|B^{1/2}(t)B^{-1/2}(\tau)\| \leq K, \qquad \|[\dot{B}^{1/2}(t) - \dot{B}^{1/2}(s)]B^{-1/2}(\tau)\| \leq L|t-s|^{\gamma}.$$

III. $h(t)$ satisfies a Hölder condition

$$\|h(t) - h(s)\| \leq C|t-s|^{\eta}, \qquad \eta \in (0, 1].$$

Let us now observe that from condition I it follows that the operator $-B^{1/2}(t)$ for each $t \in [0, T]$ is the infinitesimal generator of an analytic semigroup $\exp\{-\xi B^{1/2}(t)\}$ with exponential decrease

$$\|\exp\{-\xi B^{1/2}(t)\}\| \leq N e^{-\beta\xi}, \quad \xi \geq 0$$

(see (^{2},^{3})). Denote by $U(t, s)$

($0 \leq s \leq t \leq T$) the operator solution of the Cauchy problem $\dot{X} + B^{1/2}(t)X = 0$, $t > s$, $X(s) = I$ (see (4)), and by $V(t, s)$ ($0 \leq t \leq s \leq T$) the operator solution of the inverse Cauchy problem $\dot{X} - B^{1/2}(t)X = 0$, $t < s$, $X(s) = I$ (the substitution $t - s = \tau$ reduces this problem to the direct Cauchy problem). Let us pass to the boundary-value problem (1), (2). We shall seek its solution in the form

$$x(t) = U(t, 0)b_1 + \int_0^t U(t, s)B^{-1/2}(s)y(s)\,ds + V(t, T)b_2 + \\ + \int_t^T V(t, s)B^{-1/2}(s)y(s)\,ds. \tag{3}$$

The elements b_1 and b_2, as well as the abstract function $y(t)$, are to be determined. If $b_i \in D$, $i = 1, 2$, then $x(t)$ is strongly continuously differentiable on $[0, T]$. Differentiating (3) with respect to t, we obtain an expression for $\dot{x}(t)$. Putting in this expression and in (3) $t = 0$ and $t = T$, and then substituting $x(0)$, $x(T)$, $\dot{x}(0)$, and $\dot{x}(T)$ into (2), we obtain a system of two linear equations which permits expressing b_1 and b_2 in terms of $y(t)$:

$$\sum_{j=1}^{2} A_{ij}b_j = C_i + \int_0^T H_i(s)B^{-1/2}(s)y(s)\,ds, \qquad i = 1, 2. \tag{4}$$

Here A_{ij}, $H_i(s)$, and C_i are known linear operators, operator-functions, and elements of E, respectively.

Let now the following condition be satisfied.

IV. For any strongly continuous functions $y(t)$ on $[0, T]$ and for any C_i, $i = 1, 2$, the system (4) has a solution (b_1, b_2), with $b_i \in D$, $i = 1, 2$, if $C_i \in D$, $i = 1, 2$.

We note now that equality (3) can be twice continuously differentiated with respect to t on $(0, T)$, if $b_i \in D$, $i = 1, 2$, and $y(t)$ satisfies the weakened Hölder condition: for $0 < t \leq s < T$,

$$\|y(t) - y(s)\| \leq C_1 \frac{|t-s|^\nu}{t^\nu (T-s)^\nu}. \tag{5}$$

Assuming these conditions to be fulfilled and differentiating the expression for $\dot{x}(t)$ once more with respect to t, and taking into account (1), as well as the expressions for b_1 and b_2, we obtain integral equations for determining $y(t)$:

$$y(t) = -\frac{1}{2}h(t) + \frac{1}{2}\dot{B}^{1/2}(t)\left\{\int_0^T R(t,s)B^{-1/2}(s)y(s)\,ds + Y_1(t)a_1 + Y_2(t)a_2\right\}, \tag{6}$$

where $R(t, s)$, $Y_1(t)$, and $Y_2(t)$ are known operator-functions. Finally, we require that the following condition be satisfied:

V. The integral equation (6), for $a_i \in D$, $i = 1, 2$, has a solution $y(t)$ that is strongly continuous on $[0, T]$ and satisfies the Hölder condition (5).

Thus proved is

Theorem 1. *Let conditions I–V be fulfilled and let* $a_i \in D$, $i = 1, 2$*; then there exists a solution of problem (1), (2). It can be written in the form*

$$x(t) = \int_0^T G(t,s)h(s)\,ds + X_1(t)a_1 + X_2(t)a_2, \tag{7}$$

where $G(t, s)$ *is the Green operator-function, and* $X_i(t)$ *is the operator solution of the homogeneous equation (1), such that* $\gamma_i(X_j) = \delta_{ij} I$.

From Theorem 1 one can obtain various assertions on the existence of solutions of various boundary-value problems, and in considering con-

of concrete problems the conditions of Theorem 1 can be weakened. In particular, the following is true.

Theorem 2. *Suppose conditions I—III and the following conditions are satisfied:*

$$1)\qquad Ne^{-\alpha T} < 1, \qquad \frac{kN}{\alpha}\left[1 + \frac{(N-1)e^{-\alpha T/2}}{1 - Ne^{-\alpha T}}\right] < 1,$$

where $\alpha = \beta - 2kN$*.*

2) In the case of problem A_1*, suppose* $a_i \in D,\ i = 1, 2$*.*

3) In the case of problem A_3*, suppose* $B(0) = B(T)$*.*

Then the problems $A_i,\ i = 1, 2, 3,$ *are solvable.*

For the proof we note that the first of the inequalities in condition 1) ensures the fulfillment of condition IV, and the second that of condition V (the applicability of the contraction mapping principle to equation (6)).

Remark 1. Condition 2) of Theorem 2 can be weakened by requiring that $a_1 \in D(B^{\varepsilon/2}(0))$, $a_2 \in D(B^{\varepsilon/2}(T))$ (see (1)), where $0 < \varepsilon < 1$; in this case the second of conditions 1) is slightly complicated.

Remark 2. If condition I is replaced by the stronger requirement that for all $t \in [0, T]$ the operator $-B(t)$ be the infinitesimal generator of a strongly continuous semigroup $\exp\{-\xi B(t)\}$, so that $\|\exp\{-\xi B(t)\}\| \le Ne^{-\beta\xi}$, then $\|\exp\{-\xi B^{1/2}(t)\}\| \le Ne^{-\beta\xi}$.

Remark 3. If $N = 1$, then condition 1) of Theorem 2 becomes the requirement that $K < {}^1/_3\beta$.

The simplest case is when $B(t) \equiv B$ does not depend on t (cf. (5)). Repeating our reasoning, we arrive at system (4), where now all A_{ij} commute, and therefore one can use an analogue of Cramer’ s rule. We compute the determinant of the system Δ:

$$\Delta = \det(A_{ij}) = \Gamma_{12}[I - S(2T)] +$$

$$+ B^{1/2}\{[\Gamma_{14} + \Gamma_{23}][I + S(2T)] + 2[\Gamma_{13} + \Gamma_{24}]S(T)\} - \Gamma_{13}B[I - S(2T)],$$

where $S(t) = \exp\{-tB^{1/2}\}$, $\Gamma_{ij} = \begin{vmatrix}\gamma_{i1} & \gamma_{j1}\\ \gamma_{i2} & \gamma_{j2}\end{vmatrix}$. The other determinants are computed similarly.

Theorem 3. *Suppose* $B(t) \equiv B$*;* Δ^{-1} *exists and is bounded; conditions I and III are satisfied, as well as one of the conditions: 1)* $\Gamma_{34} \ne 0$*; 2)* $\Gamma_{34} = 0$, $\Gamma_{14} + \Gamma_{23} \ne 0$*; 3)* $\gamma_{i3} = \gamma_{i4} = 0,\ i = 1, 2,\ \Gamma_{12} \ne 0$*. Then problem (1), (2) is solvable. Its solution is given by formula (3), where* $y(t) = -{}^1/_2 h(t)$*,* (b_1, b_2) *is the solution of system (4).*

Theorem 4. *Suppose* $B(t) \equiv B$*,* $Ne^{-\beta T} < 1$ *and conditions I and III are satisfied; then the problems* $A_i,\ i = 1, 2, 3,$ *are solvable.*

The results obtained partially carry over to the equation

$$\ddot{x} + A(t)\dot{x} - B(t)x = h(t), \tag{8}$$

where $A(t)$ is a closed linear operator in E with domain dense in E, not depending on t. Suppose that, for every $t \in [0,T]$, $-A(t)$ is the infinitesimal generator of a strongly continuous group $\exp\{-\xi A(t)\}$. Introduce $\theta(t)$, the operator solution of the Cauchy problem: $\dot{\theta}(t) + {}^1/_2 A(t)\theta = 0,\ \theta(0) = I$. After the substitution $x = \theta(t)z$, equation (8) is reduced to an equation of the form (1) with operator

$$\widetilde{B}(t) = \theta^{-1}(t)\{\,{}^1/_2\dot{A}(t) + {}^1/_4 A^2(t) + B(t)\,\}\theta(t).$$

In this way we have found the solvability conditions for problem A_1 for (8).

We turn to questions of uniqueness of the solution of problems (1), (2). Consider in E^* the problem adjoint to problem (1), (2):

$$\ddot{\psi} - B^*(t)\psi = \varphi(t), \qquad \gamma_i^*(\psi) = 0, \qquad i = 1, 2. \tag{9}$$

Theorem 5. *Suppose problem (9) is solvable for arbitrary right-hand sides $\varphi(t)$ satisfying a Hölder condition of type III; then the solution of problem (1), (2) is unique.*

We note that from this follows the uniqueness of the solution of problem (1), (2), if E is a Hilbert space and the operator $B(t)$ and the boundary conditions

(2) are self-adjoint. In the general case, using the solvability conditions obtained above, it is not difficult to establish the solvability of the adjoint problem. In this way the following is proved.

Theorem 6. *Suppose that the following conditions are satisfied:*

I*. *Condition I is satisfied and, in addition, $D(B^*(t))$ is dense in E^* for each $t \in [0,T]$ (the latter condition is satisfied if E is reflexive).*

II*. *For $t, \tau \in [0,T]$ the operator $\overline{B^{-1/2}(\tau)B^{1/2}(t)}$ is continuously differentiable with respect to t in the uniform operator topology, and $B^{-1/2}(\tau)\dot{B}^{1/2}(t)$ satisfies the Hölder condition (see II).*

Then from the solvability of problem (1), (2) with self-adjoint boundary conditions there follows the uniqueness of its solution.

Let us note that the boundary conditions of the problems A_i, $i = 1, 2, 3$, are self-adjoint. In the case of problem A_1, the uniqueness theorem is valid under weaker restrictions and is a consequence of one-sided estimates and the maximum principle.

Theorem 7. *Consider the eigenvalue problem* $\ddot{x}-B(t)x=\lambda x$ *with homogeneous boundary conditions of one of the problems* A_i, $i=1,2,3$. *Suppose that the hypotheses of Theorem 2, condition I and* II^* *of Theorem 6 are satisfied, and suppose that* $B^{-1}(t)$ *is completely continuous for each* $t\in[0,T]$. *Then there exists a finite or countable set of eigenvalues; all of them are of finite multiplicity and isolated. If the set of eigenvalues is countable, then* $|\lambda_n|\to+\infty$ *as* $n\to+\infty$.

The results obtained have applications to systems of ordinary differential equations, to integro-differential equations, and to elliptic equations.

Example. $E=L_p(\Omega)$, where Ω is a simply connected bounded domain in R^n with sufficiently smooth boundary Γ,

$$B(t)x\equiv\sum_{i,j=1}^{n}a_{ij}(t,\xi)\frac{\partial^2 x}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^{n}a_i(t,\xi)\frac{\partial x}{\partial\xi_i}+a(t,\xi)x;$$

$t\in[0,T]$; $\xi=(\xi_1,\xi_n)$; a_{ij},a_i

and a are sufficiently smooth and

$$\sum_{i,j=1}^{n}a_{ij}(t,\xi)q_iq_j\geq k\sum_{i=1}^{n}q_i^2,\qquad k=\text{const}>0.$$

The domain of definition of $B(t)$ consists of the functions $\dot{W}_p^{(2)}(\Omega)$. In the cylinder $\Omega\times(0,T)$ consider the elliptic equation

$$\frac{\partial^2 x}{\partial t^2}+\sum_{i,j=1}^{n}a_{ij}(t,\xi)\frac{\partial^2 x}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^{n}a_i(t,\xi)\frac{\partial x}{\partial\xi_i}+a(t,\xi)x=h(t,\xi)$$

with boundary conditions $x\big|_{\Gamma\times[0,T]}=0$ and $\gamma_i(x)=d_i(\xi)$, $i=1,2$ (see (2)). By a solution of this problem we shall mean a solution of problem (1), (2). It is not difficult to rephrase the results obtained for this case.

The author expresses his gratitude to L. A. Lyusternik for his constant attention to the work.

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Received
4 I 1966

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Note: Figure translations are in progress. See original paper for figures.

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