SCATTERING OF A PLANE ELECTROMAGNETIC WAVE BY MULTIPLE SPHERES

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1966

SovietRxiv

Abstract

Full Text

UDC 517.946.9

SCATTERING OF A PLANE ELECTROMAGNETIC WAVE BY MULTIPLE SPHERES

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At the present time only a small number of works are known in which the problem of diffraction of a plane electromagnetic wave by several spheres is considered. Such a problem for two spheres was apparently first posed by V. Trinks [1] in 1934 in connection with the study of the scattering properties of a homogeneous medium consisting of spherical particles, and was solved by him for the case of dipole particles in the presence of axial symmetry. An exposition of V. Trinks's method as applied to two spheres with a common axis of symmetry, when the incident wave propagates in the direction of the axis of symmetry, may also be found in the work of G. Mie [2]*. In the work of O. A. Germogenova [3], Trinks' s method is extended also to the case of an arbitrary arrangement of two spheres relative to the wave vector of the incident wave and arbitrary sizes of the scattering spheres. As a special case, the problem of scattering of a plane electromagnetic wave by two spheres in the presence of axial symmetry can be obtained from [4]. Various questions in the theory of diffraction of waves by two or several spheres are considered theoretically or experimentally, for example, in the works [5–8] (see the literature review in [9], p. 503).

In the present work a rigorous solution is given of the general problem of diffraction of a plane linearly polarized electromagnetic wave

$$\mathbf{E}^i = \mathbf{E}_0 e^{ik\mathbf{nr}-i\omega t}, \qquad \mathbf{H}^i = \mathbf{H}_0 e^{ik\mathbf{nr}-i\omega t} \tag{1}$$

(the factor $\exp(-i\omega t)$ will henceforth be omitted everywhere) by several spheres forming a linear system, under the assumption that they are all perfectly conducting and that the direction of propagation of the wave, determined by the unit vector $\mathbf{n}$, forms an arbitrary angle α with the axis of rotation common to the spheres. In its mathematical formulation this problem reduces to the solution of two exterior boundary-value problems for the scalar Helmholtz wave equation. The exact formulas and the approximate formulas obtained from them, by which the desired electromagnetic fields are described, are derived by using a mathematical apparatus generally different from that used in [1, 3], although at the same time a certain commonality of the ideas underlying the applied method of solving the problem is preserved. Its justification is also given.

1. Formulation of the problem. Let the Cartesian coordinate system $Oxyz$

Fig. 1

Figure 1: Fig. 1

and the local coordinate systems $O_j x_j y_j z_j$ be associated with the spheres in such a way that

* In [2] some experimental results are also given, pertaining to the case when the wave vector is directed at an angle to the line of centers of the spheres.

as shown in Fig. 1 (each sphere is assigned its own index j, $j = 1, 2, \dots, N$). In addition, local spherical coordinates r_j, θ_j, φ_j ($j = 1, \dots, N$) are assigned to each sphere, in which the surface of the j-th sphere is given by the equation $r_j = a_j$. The distance between the centers of two spheres, for example between the j-th and the ν-th, is denoted by $l_{j\nu}$. The axis Oz of the system $Oxyz$ is the axis of symmetry of the spheres. The axes $O_j x_j$ ($j = 1, 2, \dots, N$) are directed so that they all lie in one plane $\varphi = 0$, coinciding with the plane of propagation of the wave (the vector $\mathbf{n}$ belongs to the plane $\varphi = 0$). It is obvious that, with the indicated introduction of coordinate systems, $\varphi_j = \varphi$ for all j. It is also obvious that if $\mathbf{r}_{0j}$ is the radius vector joining O with the origin of the j-th coordinate system (the point O_j), then in the coordinates of the j-th sphere the wave (1) can be written in the form

$$\mathbf{E}^i = \mathbf{E}_0 e^{ik\mathbf{n}\mathbf{r}_{0j} + ik\mathbf{n}\mathbf{r}_j}, \qquad \mathbf{H}^i = \mathbf{H}_0 e^{ik\mathbf{n}\mathbf{r}_{0j} + ik\mathbf{n}\mathbf{r}_j}, \tag{2}$$

where $\mathbf{r}_j$ is the radius vector joining an arbitrary observation point with O_j ($j = 1, 2, \dots, N$). In what follows it is assumed that the electric vector $\mathbf{E}^i$ of the field of the incident wave makes an angle β with the plane $\varphi = 0$.

Fig. 1

In spherical coordinates the total electromagnetic field $\mathbf{E}, \mathbf{H}$, considered as the sum of the field of the incident wave $\mathbf{E}^i, \mathbf{H}^i$ and the field $\mathbf{E}^s, \mathbf{H}^s$ scattered by the spheres, can be found, as is known [10, 11], from the relations

$$\mathbf{E} = \operatorname{rot}\operatorname{rot}(\mathbf{r}\Pi) + i\omega\mu \operatorname{rot}(\mathbf{r}\Pi^*),$$

$$\mathbf{H} = \operatorname{rot}\operatorname{rot}(\mathbf{r}\Pi^*) - i\omega\varepsilon \operatorname{rot}(\mathbf{r}\Pi), \qquad \mathbf{r} = r \cdot \mathbf{i}_r, \quad |\mathbf{r}| = r \tag{3}$$

(μ, ε are the physical constants of the medium outside the spheres; $k = \omega\sqrt{\mu\varepsilon}$), if the electric and magnetic Debye potentials of the total field, which we shall denote by Π and Π^*, respectively, have first been found (they are related to the only nonzero radial components, in spherical coordinates, of the electric and magnetic * Hertz vectors by the relations $\Pi = \Pi_r / r$, $\Pi^* = \Pi_r^* / r$). Since the

Debye potentials for a given incident-wave field may be regarded as known (the form of these potentials, denoted by Π^i and Π^{*i}, is given below), it is obvious that the problem of finding the vectors $\mathbf{E}, \mathbf{H}$ will be solved if the potentials Π^s, Π^{*s} of the field $\mathbf{E}^s, \mathbf{H}^s$ scattered by the spheres are found. The latter must be solutions of the equations

$$\Delta\Pi^s + k^2\Pi^s = 0, \qquad \Delta\Pi^{*s} + k^2\Pi^{*s} = 0, \tag{4}$$

satisfying, on the surface of each sphere, simultaneously the boundary conditions

$$\frac{\partial}{\partial r_j}\left[r_j(\Pi^i + \Pi^s)\right] = 0, \qquad \Pi^{*i} + \Pi^{*s} = 0 \quad (r_j = a_j, \quad j = 1, 2, \dots, N). \tag{5}$$

In addition, Π^s and Π^{*s} must satisfy the radiation conditions at infinity.

After solving problem (4)–(5), the components of the field $\mathbf{E}, \mathbf{H}$ are found from the equations

$$E_r = \frac{\partial^2(r\Pi)}{\partial r^2} + k^2(r\Pi), \qquad H_r = \frac{\partial^2(r\Pi^*)}{\partial r^2} + k^2(r\Pi^*),$$

$$E_\theta = \frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r\partial\theta} + \frac{i\omega\mu}{r\sin\theta}\frac{\partial(r\Pi^*)}{\partial\varphi},$$

$$H_\theta = \frac{1}{r}\frac{\partial^2(r\Pi^*)}{\partial r\partial\theta} - \frac{i\omega\varepsilon}{r\sin\theta}\frac{\partial(r\Pi)}{\partial\varphi}, \tag{6}$$

$$E_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(r\Pi)}{\partial r\partial\varphi} - \frac{i\omega\mu}{r}\frac{\partial(r\Pi^*)}{\partial\theta},$$

$$H_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(r\Pi^*)}{\partial r\partial\varphi} + \frac{i\omega\varepsilon}{r}\frac{\partial(r\Pi)}{\partial\theta},$$

which are obtained from (3) if the latter are written in one of the coordinate systems.

2. Solution of the problem. We shall seek the solution of equations (4) in the form of a sum of N expansions:

$$\Pi^s = \sum_{j=1}^{N}\Pi_j, \tag{7}$$

$$\Pi^{*s} = \sum_{n=1}^{N} \Pi_j^*, \tag{8}$$

where

$$\Pi_j = \frac{E_0}{ik} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{mn}^j \, h_n^{(1)}(kr_j) \, P_n^m(\cos\theta_j) e^{im\varphi}, \tag{9}$$

$$\Pi_j^* = \frac{E_0 \sqrt{\dfrac{\varepsilon}{\mu}}}{ik} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_{mn}^j \, h_n^{(1)}(kr_j) \, P_n^m(\cos\theta_j) e^{im\varphi} \tag{10}$$

with coefficients a_{mn}^j, b_{mn}^j, which are to be determined subsequently from the boundary conditions (5) (obviously, for each j the series (9), (10), with the correspondingly determined coefficients, express the field scattered by a single sphere). To find a_{mn}^j, b_{mn}^j, we represent the Debye potentials Π^i, Π^{*i} of the incident wave in the form of expansions analogous to (9), (10), in terms of the proper spherical wave functions of the j-th sphere. For this purpose, along with the local system $O_j x_j y_j z_j$, we introduce one more coordinate system $O_j x_j' y_j' z_j'$, also associated with the j-th sphere, whose axis $O_j z_j'$ is directed along the vector $\mathbf{n}$, while the axes $O_j x_j'$, $O_j y_j'$ form, with the axes $O_j x_j$, $O_j y_j$ of the system $O_j x_j y_j z_j$, the Euler angles

$$\frac{\pi}{2} - \beta \quad \text{and} \quad \frac{3\pi}{2}$$

respectively (the angle between $O_j z_i$ and $O_j z_j'$ is equal to α). It is assumed here that, in the system $O_j x_j' y_j' z_j'$,

$$E_{x_j'}^{i'} = \sqrt{\frac{\mu}{\varepsilon}} \, H_{y_j'}^{i'} = E_0 e^{ik\mathbf{n}\mathbf{r}_{0j}} e^{ikz_j'},$$

and

$$E_{y_j'}^{i'} = E_{z_j'}^{i'} = H_{x_j'}^{i'} = H_{z_j'}^{i'} = 0,$$

as a result of which, in the spherical coordinates $r_j', \theta_j', \varphi_j'$ $(r_j' = r_j)$, associated with the system $O_j x_j' y_j' z_j'$, the Debye potentials Π^i, Π^{*i} of the incident-wave field are given by the expansions [11–13]:

$$\Pi^i = \frac{E_0}{ik} e^{ik\mathbf{n}\mathbf{r}_{0j}} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n(kr_j) P_n^1(\cos\theta_j') \cos\varphi_j', \tag{11}$$

$$\Pi^{*i} = \frac{E_0}{ik}\sqrt{\frac{\varepsilon}{\mu}}\, e^{ik\mathbf{n}\mathbf{r}_{0j}} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n(kr_j) P_n^1(\cos\theta_j') \sin\varphi_j'. \tag{12}$$

In order now to make use of the orthogonality property of the spherical functions on the surface of the j-th sphere in the boundary conditions (5), it is necessary first to write the expansions (11), (12) in the coordinates r_j, θ_j, φ. For this purpose one needs the addition formulas for the spherical functions $P_n^m(\cos\theta)\cos(m\varphi)$, $\sin(m\varphi)$. They can be obtained as special cases of the addition formula of general form, given in [14, 26] for generalized spherical functions of the n-th order T_{km}^n:

$$T_{km}^n(\psi_1, \theta_1, \gamma_1) = \sum_{l=-n}^{n} T_{kl}^n(\psi_2, \theta_2, \gamma_2) T_{lm}^n(\psi_3, \theta_3, \gamma_3). \tag{13}$$

Here T_{km}^n is defined by the relation

$$T_{km}^n(\psi, \theta, \varphi) = e^{-ik\psi} P_{km}^n(\theta) e^{-im\varphi}, \tag{14}$$

where, in turn,

$$P_{km}^n(\theta) = \frac{(-1)^{n-k} i^{m-k}}{2^n (n-k)!} \sqrt{\frac{(n-k)!(n+m)!}{(n+k)!(n-m)!}}\, (1-\mu)^{-\frac{m-k}{2}} \times (1+\mu)^{-\frac{m+k}{2}} \frac{d^{n-m}}{d\mu^{n-m}} \left[(1-\mu)^{n-k}(1+\mu)^{n+k}\right], \tag{15}$$

$$(\mu = \cos\theta)$$

and where the function $P_{km}^n(\theta)$ for $k = 0$ is expressed through the associated Legendre function $P_n^m(\cos\theta)$ by the equality

$$P_{0m}^n(\theta) = P_{0,-m}^n(\theta) = i^{-m} \sqrt{\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta). \tag{16}$$

The generalized spherical functions T_{km}^n are the elements of the matrix $\|T_{km}^n(\psi, \theta, \varphi)\|$ $(k, m = -n, \dots, n)$. They are defined on the surface of the sphere and belong to the space of functions in which the irreducible representation of weight n of the rotation group of three-dimensional space (the sphere) is realized; ψ, θ, φ are Euler angles. In what follows we shall need the following properties of the generalized spherical functions:

$$P^n_{km}(\theta) = \begin{cases} 0, & \text{if } k \neq m, \\ 1, & \text{if } k = m, \end{cases} \qquad \mu = \cos\theta, \quad \theta = 0, \tag{17}$$

$$P^n_{km}(\theta) = (-1)^n, \quad k = -m, \quad \mu = \cos\theta, \quad \theta = \pi.$$

In addition,

$$T^n_{mm}(0,0,0) = 1, \qquad T^n_{-k,-m}(\psi,\theta,\varphi) = T^n_{km}(-\psi,\theta,-\varphi),$$

$$P^n_{km}(\theta) = P^n_{mk}(\theta).$$

If in (13) we put $k = 0$ and use (14), (16), then, as applied to our problem, we find that

$$P^m_n(\cos\theta'_j)e^{-im\varphi'_j} = i^m\sqrt{\frac{(n+m)!}{(n-m)!}}\times$$

$$\times\sum_{l=-n}^{n} T^n_{-l,m}(\psi_1,\theta_1,\varphi_1)(-i)^l\sqrt{\frac{(n-l)!}{(n+l)!}}\,P^l_n(\cos\theta_j)e^{il\varphi} \tag{18}$$

or, if we use the relation [15],

$$P^{-m}_n(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P^m_n(x), \tag{19}$$

$$P^m_n(\cos\theta'_j)e^{im\varphi'_j} = i^m\sqrt{\frac{(n+m)!}{(n-m)!}}\sum_{l=-n}^{n} T^n_{-l,-m}(\psi_1,\theta_1,\varphi_1)\times$$

$$\times(-i)^l\sqrt{\frac{(n-l)!}{(n+l)!}}\,P^l_n(\cos\theta_j)e^{il\varphi}, \tag{20}$$

where

$$\psi_1 = \frac{\pi}{2} - \beta, \qquad \theta_1 = \alpha, \qquad \varphi_1 = \frac{3\pi}{2}.$$

From (18), (20) we obtain that

$$P_n^m(\cos\theta_j')\cos m\varphi_j' = \sum_{l=-n}^{n} L_{lm}^n P_n^l(\cos\theta_j)e^{il\varphi},$$

(21)

$$P_n^m(\cos\theta_j')\sin m\varphi_j' = \sum_{l=-n}^{n} L_{lm}^{*n} P_n^l(\cos\theta_j)e^{il\varphi},$$

where we have set

$$L_{lm}^n = \frac{1}{2} i^{m-l}\left\{\frac{(n+m)!(n-l)!}{(n-m)!(n+l)!}\right\}^{1/2}\left(T_{-l,-m}^n + T_{-l,m}^n\right), \tag{22}$$

$$L_{lm}^{*n} = \frac{i^{m-l-1}}{2}\left\{\frac{(n+m)!(n-l)!}{(n-m)!(n+l)!}\right\}^{1/2}\left(T_{-l,-m}^n - T_{-l,m}^n\right). \tag{23}$$

On the basis of (21), the expansions (11), (12) in the coordinates r_j, θ_j, φ are written in the form

$$\Pi^i = \frac{E_0 e^{ik\mathbf{n}\mathbf{r}_{0j}}}{ik}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n \frac{2n+1}{n(n+1)} L_{m1}^n j_n(kr_j) P_n^m(\cos\theta) e^{im\varphi}, \tag{24}$$

$$\Pi^{*i} = \frac{E_0\sqrt{\dfrac{\varepsilon}{\mu}}\, e^{ik\mathbf{n}\mathbf{r}_{0j}}}{ik}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n \frac{2n+1}{n(n+1)} L_{m1}^{*n} j_n(kr_j) P_n^m(\cos\theta_j) e^{im\varphi}. \tag{25}$$

In order to write the wave functions $h_n^{(1)}(kr_\nu)P_n^m(\cos\theta_\nu)$, which enter into (7), (8) and are written in the coordinates of the ν-th sphere, in the coordinates of the j-th sphere, we apply the addition theorem for spherical wave functions

$$h_n^{(1)}(kr_\nu)P_n^m(\cos\theta_\nu) = \sum_{q=0}^{\infty} Q_{mqmn}(r_{\nu j},\theta_{\nu j}) j_q(kr_j) P_q^m(\cos\theta_j), \quad r_{\nu j} > r_j, \tag{26}$$

where

$$Q_{mqmn} = \frac{2i^{q-n}}{N_{mq}}\sum_{\sigma=|n-q|}^{n+q} i^\sigma b_\sigma^{(qmnm)} h_\sigma^{(1)}(kl_{\nu j}) P_\sigma(\cos\theta_{\nu j}), \quad l_{\nu j} = r_{\nu j} \tag{27}$$

($r_{\nu j}, \theta_{\nu j}$ are the spherical coordinates of the point O_j in the coordinate system with origin at the point O_ν). Formula (26) can be obtained as a special case of formula (13) in [16]* when the spheroidal coordinates there degenerate into spherical ones, when $c \to 0$. Here b_σ denotes the expansion coefficients [16]

$$P_q^m(\cos\theta) P_n^m(\cos\theta) = \sum_{\sigma=|q-n|}^{q+n} b_\sigma^{(qmnm)} P_\sigma(\cos\theta), \tag{28}$$

where

$$b_\sigma^{(qmnm)} = (-1)^m \sqrt{\frac{(q+m)!(n+m)!}{(q-m)!(n-m)!}} (qn00 \mid \sigma 0)(qnm, -m \mid \sigma 0), \tag{29}$$

where $(n_1 n_2 m_1 m_2 \mid nm)$, $n = n_1 + n_2$, $m = m_1 + m_2$, is the symbolic notation for the Clebsch–Gordan coefficients, whose explicit form is given, for example, in [14, 20] (various forms of their expression can be found in [21]). On the basis of (26), the series for Π_ν, Π_ν^* from (7), (8) are written in the coordinates of the j-th sphere in the form

$$\Pi_\nu = \frac{E_0}{ik} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{\infty} a_{nm}^\nu Q_{mqmn} j_q(kr_j) P_q^m(\cos\theta_j) e^{im\varphi}, \tag{30}$$

$$\Pi_\nu^* = \frac{E_0 \sqrt{\dfrac{\varepsilon}{\mu}}}{ik} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{\infty} b_{nm}^\nu Q_{mqmn} j_q(kr_j) P_q^m(\cos\theta_j) e^{im\varphi}. \tag{31}$$

* In another form, the addition theorem is given, for example, in [18–19]. It can also be obtained from [17].

If we now satisfy the boundary conditions (5), then, on the basis of (24), (25), (7) –(10) and (30), (31), as well as the orthogonality property of spherical functions on the surface of the j-th sphere, for the coefficients a_{mn}^j, b_{mn}^j we obtain infinite systems of linear equations

$$a_{mn}^j + \sum_{\substack{\nu=1 \\ \nu \neq j}}^{N} \sum_{q=0}^{\infty} \alpha_{mnq}^{\nu j} a_{mq}^\nu = f_{mn}^j, \tag{32}$$

$$(|m| \leq n; \quad j = 1, 2, \dots, N),$$

$$b_{mn}^{j}+\sum_{\substack{\nu=1\\ \nu\neq j}}^{N}\sum_{q=0}^{\infty}\beta_{mnq}^{\nu j}b_{mq}^{\nu}=\varphi_{mn}^{j}, \tag{33}$$

where

$$\alpha_{mnq}^{\nu j}=\frac{\psi_n'(ka_j)}{\xi_n^{(1)\prime}(ka_j)}\,Q_{mnmq}, \qquad \beta_{mnq}^{\nu j}=\frac{\psi_n(ka_j)}{\xi_n^{(1)}(ka_j)}\,Q_{mnmq}, \tag{34}$$

$$f_{mn}^{j}=-e^{i\mathbf{knr}_{0j}}i^{n}\frac{2n+1}{n(n+1)}\,L_{m1}^{n}\,\frac{\psi_n'(ka_j)}{\xi_n^{(1)\prime}(ka_j)}, \tag{35}$$

$$\varphi_{mn}^{j}=-e^{i\mathbf{knr}_{0j}}i^{n}\frac{2n+1}{n(n+1)}\,L_{m1}^{*n}\,\frac{\psi_n(ka_j)}{\xi_n^{(1)}(ka_j)}. \tag{36}$$

Here, in Debye’ s notation,

$$\psi_n(x)=xj_n(x)=\sqrt{\frac{2x}{\pi}}\,J_{n+1/2}(x), \tag{37}$$

$$\xi^{(1)}(x)=xh_n^{(1)}(x)=\sqrt{\frac{2x}{\pi}}\,H_{n+1/2}^{(1)}(x). \tag{38}$$

If in (32), (33) we make a change of unknowns, putting

$$a_{mn}^{j}=\psi_n'(ka_j)A_{mn}^{j}, \qquad b_{mn}^{j}=\psi_n(ka_j)B_{mn}^{j}, \tag{39}$$

then, after substituting (39) into (32), (33), for the new unknowns A_{mn}^{j} and B_{mn}^{j} we obtain the systems

$$A_{mn}^{j}+\sum_{\substack{\nu=1\\ \nu\neq j}}^{N}\sum_{q=0}^{\infty}\gamma_{mnq}^{\nu j}A_{mq}^{\nu}=F_{mn}^{j}, \tag{40}$$

$$B_{mn}^{j}+\sum_{\substack{\nu=1\\ \nu\neq j}}^{N}\sum_{q=0}^{\infty}\delta_{mnq}^{\nu j}B_{mq}^{\nu}=\Phi_{mn}^{j}, \tag{41}$$

where

$$\gamma_{mnq}^{\nu j}=\psi_q'(ka_\nu)\alpha_{mnq}^{\nu j}/\psi_n'(ka_j), \qquad F_{mn}^{j}=f_{mn}^{j}/\psi_n'(ka_j), \tag{42}$$

$$\delta^{\nu j}_{mnq} = \psi_q(ka_\nu)\beta^{\nu j}_{mnq}/\psi_n(ka_j), \qquad \Phi^j_{mn} = \varphi^j_{mn}/\psi_n(ka_j), \tag{43}$$

which, for any value of $|m| \le n$, are quasiregular and are uniquely solvable by the method of reduction [22]. The proof of this assertion can be carried out according to the same scheme as ···

for example, in [4]. Indeed, it is not difficult to show that, for any $0 \le m \le n$, the elements of the matrix of system (41) (since systems (40) and (41) have, in principle, the same structure, it will suffice below to consider one of them, for example (41)) satisfy the inequality

$$\left|\delta^{vj}_{mnq}\right| < C\,\frac{(n+q)!}{(q+m)!(n-m)!}\left(\frac{a_j}{l_{vj}}\right)^{n-m}\left(\frac{a_v}{l_{vj}}\right)^{q+m}, \quad C = \text{const}, \tag{44}$$

and the right-hand sides of the inequality

$$\left|\Phi^j_{mn}\right| < \text{const}\, g^n, \qquad 0 < g < 1 \tag{45}$$

(the case of negative m is discussed below). Thus, turning to (27), we find that

$$\left|Q_{mnmq}\right| \le \frac{(2q+1)!(q-m)!}{(q+m)!}\sum_{\sigma=|n-q|}^{n+q}|b_\sigma|\cdot|h^{(1)}_\sigma(kl_{vj})|,$$

where, as follows from (28),

$$b_\sigma = \frac{2\sigma+1}{2}\int_0^\pi P^m_q(\cos\theta)\,P^m_n(\cos\theta)\,P_\sigma(\cos\theta)\,\sin\theta\,d\theta,$$

or, if the mean-value theorem is applied here,

$$b_\sigma = \frac{2\sigma+1}{2}\,\pi P^m_q(\cos\theta^*)P^m_n(\cos\theta^*)P_\sigma(\cos\theta^*)\sin\theta^*, \tag{46}$$

where $0 < \theta^* < \pi$. Since, for $0 < \theta^* < \pi$,

$$|P^{\pm m}_n(\cos\theta^*)| \le \text{const}\,\frac{(n\pm m)!}{\sqrt{n}\,n!} \tag{47}$$

(if $n \ge 1,\ n-m+1 > 0,\ m \ge 0$, [23, 15]), it follows from (46) that

$$|b_\sigma| \le \text{const}_1(2\sigma+1)\frac{(n+m)!(q+m)!}{\sqrt{\sigma n q}\,n!q!}, \tag{48}$$

and therefore

$$\left|Q_{mnmq}\right| < \text{const}_2 \frac{(2q+1)(q-m)!(n+m)!\sqrt{n+q}}{\sqrt{nq}\, n!q!} \left|h^{(1)}_{n+q}(kl_{vj})\right|, \qquad (49)$$

since

$$\sum_{\sigma=|n-q|}^{n+q} \frac{2\sigma+1}{\sqrt{\sigma}} \left|h^{(1)}_{\sigma}(kl_{vj})\right| < \text{const}_3 \sqrt{n+q} \left|h^{(1)}_{n+q}(kl_{vj})\right|$$

(see, for example, [4]). Now using the asymptotic formulas, with respect to n, for the spherical Bessel functions $j_n(x)$, $h^{(1)}_n(x)$, and Stirling' s formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, by the same method as in [4] we find that, for all n and q and any $0 \le m \le n, q$, inequality (44) holds (taking into account here that $n! \le \text{const}_4 \sqrt{n}\, n^n e^{-n}$ for all $n \ge 0$).

From the unitarity of the matrix $\|T^n_{km}\|$ it follows that [14]

$$\sum_{m=-n}^{n} |T^n_{km}|^2 = \sum_{m=-n}^{n} |D^n_{km}|^2 = 1 \qquad (k = -n, \dots, n), \qquad (50)$$

whence, in turn, follows the boundedness in modulus of its elements—for all n and m, $k = -n, \dots, n$, and, obviously, $|T^n_{km}| \leqslant 1$. Therefore, for the quantity L^{*n}_{ml} from (36) one obtains the inequality

$$|L^{*n}_{ml}| \leqslant \left\{ \frac{(n+1)!(n-m)!}{(n-1)!(n+m)!} \right\}^{1/2}, \qquad (50')$$

on the basis of which

$$|\Phi^j_{mn}| \leqslant \text{const}_5 \frac{(2n+1)\left\{ \dfrac{(n-m)!}{(n+m)!} \right\}^{1/2}}{\sqrt{n(n+1)} \left|h^{(1)}_n(ka_j)\right|}. \qquad (51)$$

Considering first the right-hand side of (51) for sufficiently large n, for which the asymptotic formula for the function $h^{(1)}_n(x)$ and Stirling' s formula for the factorials are valid, we then find that the inequality (45) is already satisfied for all n and $0 \leqslant m \leqslant n$.

In the case of negative values of m ($0 < -m \leqslant n$), in system (41) replace m by $-m'$, where now $m' > 0$. Then, on the basis of ($57'$) given below, from (41) we have

$$B^j_{-m',n} + \sum_{\substack{\nu=1\\ \nu\neq j}}^{N} \sum_{q=0}^{\infty} \delta^{\nu j}_{m'nq} \frac{(q-m')!(n+m')!}{(q+m')!(n-m')!} B^{\nu}_{-m',q} = \Phi^j_{-m',n}$$

or

$$\overline{B}^{\,j}_{m'n} + \sum_{\substack{\nu=1\\ \nu\neq j}}^{N} \sum_{q=0}^{\infty} \delta^{\nu j}_{m'nq}\, \overline{B}^{\,\nu}_{m'q} = \overline{\Phi}^{\,j}_{m'n}, \tag{51'}$$

where

$$\overline{B}^{\,j}_{m'n} = \frac{(n-m')!}{(n+m')!} B^j_{-m',n}, \qquad \overline{\Phi}^{\,j}_{m'n} = \frac{(n-m')!}{(n+m')!} \Phi^j_{-m',n}. \tag{51''}$$

The matrix of coefficients of system $(51')$ is the same as that of system (41). It is not difficult to verify, on the basis of $(51'')$ and $(50')$, that $\overline{\Phi}^{\,j}_{m'n} \in l^2$. Therefore all the further arguments connected with system (41), considered for $m \geqslant 0$, will apply equally to system $(51')$, and thereby to the case of negative values of m in (41).

The matrix composed of the right-hand sides (44) with respect to n, q obviously forms a completely continuous form in the Hilbert space l^2, since for it the condition

$$\sum_{n,q}^{\infty} \left[\frac{(n+q)!}{(q+m)!(n-m)!} \left(\frac{a_j}{l_0}\right)^{n-m} \left(\frac{a_\nu}{l_0}\right)^{q+m} \right]^2 < \infty, \qquad l_0 = \min_{\nu,j} l_{\nu j}, \tag{52}$$

is satisfied if one assumes that $l_0 > a_\nu + a_j$, i.e., that the spheres with numbers ν and j $(\nu, j = 1, 2, \ldots, N)$ do not touch. The right-hand sides of (45) are elements of the space l^2, since $\sum_n^\infty |g^n|^2 < \infty$. Therefore, if instead of system (41) one considers the system with coefficient matrix composed of the right-hand sides (44) and with free terms having the form of the right-hand side of (45), then Hilbert' s alternative on the solvability of infinite systems of linear equations is applicable to it: such a system is either uniquely solvable and its solution belongs to l^2, or it has nontrivial solutions belonging to l^2, corresponding to it

homogeneous system [22]. If such a system is taken as a majorant for system (41), then, on the basis of the comparison theorems for infinite systems, this Hilbert proposition can also be carried over to system (41). It is further evident that the homogeneous system corresponding to the inhomogeneous system (41) cannot have nontrivial solutions, since the assumption of their existence would in turn mean the existence of eigenoscillations in the exterior domain in the presence of the radiation conditions, which, as is known (see, for example, p. 171, and also the translator' s editorial note [11] on p. 224), contradicts the unique solvability of the exterior boundary-value problem in the case of a real parameter k. Thus, from the above it follows that system (41) is uniquely solvable and its solution, for any $0 \le m \le n$, will satisfy the condition

$$\sum_{n}^{\infty} |B^j_{mn}|^2 < \infty.$$

Since the majorizing system is quasiregular, (41) is also a quasiregular system and its solution can be found by the method of reduction with any prescribed accuracy.

Without any changes in the reasoning, the solvability by the method of reduction is proved also for system (40), whose solution satisfies the condition

$$\sum_{n}^{\infty} |A^j_{mn}|^2 < \infty$$

uniformly with respect to $0 \le m \le n$.

If we now turn to the series (9), (10) and make there the substitution (39), then, on the basis of (47) and the fact established above that the coefficients*) $A^j_{mn}, A^j_{m'n}, B^j_{mn}, \overline{B}^j_{m'n}$ are uniformly bounded with respect to n and $0 \le m, m' \le n$, it is easy to prove the absolute and uniform convergence of the series (9), (10) and of the series obtained from them by differentiation with respect to r, θ, φ at any point outside the spheres, where $r_j > a_j$. This proves the existence of a solution of problem (4)–(5) in the form (7), (8), which is unique by virtue of the uniqueness of the determination of the coefficients $A^j_{mn}, B^j_{mn}, A^j_{m'n}, \overline{B}^j_{m'n}$.

To obtain numerical results, owing to the uniform convergence of the series (9), (10), the required computational accuracy can be achieved if in (9), (10) only a finite number of the first terms (with respect to n) is retained, depending in each case on the value of the parameter ka. In exactly the same way, in solving systems (40), (41), owing to their solvability by the method of reduction and the convergence of the computational process, for the computation with the required degree of accuracy of the coefficients A^j_{mn}, B^j_{mn} in each case (for each $|m| \le n$) one in fact assumes finiteness of the values of n and q. On the basis of these considerations it appears possible to apply in (34), for sufficiently large $l = \min l_{\nu j}$, the formula asymptotic with respect to $kl_{\nu j}$,

$$h^{(1)}_{\sigma}(kl_{\nu j}) = (-i)^{\sigma+1} e^{ikl_{\nu j}} / kl_{\nu j} \tag{53}$$

under the assumption that $kl_{\nu j} \gg 1$, $kl_{\nu j} \gg \sigma$, where $|n - q| \le \sigma \le n + q$, and then, in the approximation under consideration,

$$Q_{mnmq} = \frac{e^{ikl_{\nu j}}}{kl_{\nu j}} \frac{2i^{\,n-q-1}}{N_{mn}} \sum_{\sigma} b_{\sigma}^{(nmqm)} P_{\sigma}(\cos\theta_{\nu j}) =$$

*) $\overline{A}^{j}_{m'n}$ are related to A^{j}_{mn} $(-m = m',\ m' > 0)$ by the relation

$$\overline{A}^{j}_{m'n} = \frac{(n-m')!}{(n+m')!} A^{j}_{-m'n}.$$

$$= \begin{cases} 0, & m \neq 0, \\ \dfrac{e^{ikl_{vj}}}{kl_{vj}} \dfrac{2i^{\,n-q-1}}{N_{mn}} P_n^m(\cos\theta_{vj}) P_q^m(\cos\theta_{vj}), & m = 0, \end{cases} \tag{54}$$

since, on the basis of (28),

$$\sum_{\sigma} b_{\sigma}^{(nmqm)} P_{\sigma}(\cos\theta_{vj}) = P_n^m(\cos\theta_{vj}) P_q^m(\cos\theta_{vj}),$$

where $\theta_{vj} = 0$ or π, while $P_n^m(\pm 1) = 0$ if $m \neq 0$. In this case the solutions of systems (40), (41) will approximately be given by the formulas

$$A^{j}_{mn} = F^{j}_{mn}, \qquad B^{j}_{mn} = \Phi^{j}_{mn}, \qquad m \neq 0. \tag{55}$$

For $m = 0$, $F^{j}_{0n} \equiv 0$ with respect to n, since, on the basis of (16), (22), in this case

$$L^{n}_{01} = \frac{i}{2}\sqrt{n(n+1)}(T^{n}_{0,-1} + T^{n}_{01}) =$$

$$= \frac{i}{2}\sqrt{n(n+1)} P^{n}_{01}(e^{i\tau} + e^{-i\tau}) = 0 \quad (\tau = 3\pi/2),$$

and then, by virtue of the absence of nontrivial solutions of the homogeneous system corresponding to the inhomogeneous system (40), $A^{j}_{0n} = 0$. From (23) we find that for $m = 0$

$$L^{*n}_{01} = \frac{1}{2}\sqrt{(n+1)n}\, P^{n}_{01}(e^{i\tau} - e^{-i\tau}) = -i\sqrt{n(n+1)} P^{n}_{01} =$$

$$= -P^{1}_{n}(\cos\alpha).$$

Therefore, to determine B^{j}_{0n} one should solve system (41), where approximately

$$\delta_{0nq}^{vj} = \frac{2i^{\,n-q-1}}{N_{0n}} \frac{e^{ikl_{vj}}}{kl_{vj}} \frac{\psi_q(ka_v)}{\zeta_n^{(1)}(ka_j)} P_n(\cos\theta_{vj}) P_q(\cos\theta_{vj}),$$

$$\Phi_{0n}^{j} = e^{iknr_{0j}} \frac{i^n(2n+1)}{n(n+1)} \frac{P_n^1(\cos\alpha)}{\zeta_n^{(1)}(ka_j)}.$$

Systems (40) and (41) are also considerably simplified in the case when the wave incident on the spheres propagates along the axis of symmetry of the spheres. In this case $\alpha = 0$ or $\alpha = \pi$, and therefore, on the basis of (17), the right-hand sides of systems (40) and (41) will be identically equal to zero for all $m \neq \pm 1$, and then

$$A_{mn}^{j} = B_{mn}^{j} = 0 \quad (m \neq \pm 1).$$

For definiteness, let $\alpha = 0$. Then in the system $O_j x_j y_j z_j$ the Debye potentials of the field of the incident wave are written in the form:

$$\Pi^i = \frac{E_0}{ik} e^{iknr_{0j}} \cos(\varphi - \beta) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n(kr_j) P_n^1(\cos\theta_j),$$

$$\Pi^{*j} = \frac{E_0}{ik} \sqrt{\frac{\varepsilon}{\mu}} e^{ik\mathbf{nr}_{0j}} \sin(\varphi - \beta) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times j_n(kr_j) P_n^1(\cos\theta_j).$$

It is not difficult to show that in this case the following relation exists between the coefficients $A_{-1,n}^{j}$ and A_{1n}^{j}, as well as $B_{-1,n}^{j}$ and B_{1n}^{j}:

$$A_{-1,n}^{j} = -A_{1n}^{j} \frac{(n+1)!}{(n-1)!} e^{i2\beta}, \tag{56}$$

$$B_{-1,n}^{j} = B_{1n}^{j} \frac{(n+1)!}{(n-1)!} e^{i2\beta}. \tag{57}$$

Indeed, from (28) and (19) it follows that

$$b_g^{(qmnn)} = \frac{(q+m)!(n+m)!}{(q-m)!(n-m)!} b_g^{(q,-m,n,-m)},$$

and then from (27) we find that

$$Q_{-m,n,-m,q} = \frac{(q-m)!(n+m)!}{(q+m)!(n-m)!}\, Q_{mnmq}, \tag{57'}$$

whence, for $m = 1$,

$$Q_{-1,n,-1,q} = \frac{(q-1)!(n+1)!}{(q+1)!(n-1)!}\, Q_{1n1q}.$$

Since

$$L^n_{11} = \frac{1}{2} e^{-i\beta}, \qquad L^n_{-1,1} = -\frac{1}{2}\frac{(n+1)!}{(n-1)!} e^{i\beta}, \tag{58}$$

$$L^{*n}_{11} = \frac{i^{-1}}{2} e^{-i\beta}, \qquad L^{*n}_{-1,1} = \frac{i^{-1}}{2}\frac{(n+1)!}{(n-1)!} e^{i\beta},$$

then, on the basis of (35), (36), and (58), we have

$$F^j_{-1,n} = -\frac{(n+1)!}{(n-1)!} e^{2i\beta} F^j_{1n}, \qquad \Phi^j_{-1,n} = \frac{(n+1)!}{(n-1)!} e^{2i\beta} \Phi^j_{1n}.$$

Therefore, turning to the systems (40), (41), putting there $m = 1,\ \alpha = 0$, after dividing both sides by $-\frac{(n+1)!}{(n-1)!} e^{2i\beta}$ and $\frac{(n+1)!}{(n-1)!} e^{2i\beta}$, respectively, we obtain that

$$\left[-\frac{(n-1)!}{(n+1)!} e^{-2i\beta} A^j_{-1,n}\right] +$$

$$+ \sum_{\substack{l=1 \\ \nu\neq j}}^{N} \sum_{q=1}^{\infty} \gamma^{\nu j}_{lq1n} \left[-\frac{(q-1)!}{(q+1)!} e^{-2i\beta} A^{\nu}_{-1,q}\right] = F^j_{1n},$$

$$\left[\frac{(n-1)!}{(n+1)!} e^{-2i\beta} B^j_{-1,n}\right] + \tag{59}$$

$$+ \sum_{\substack{\nu=1 \\ \nu\neq j}}^{N} \sum_{q=1}^{\infty} \delta^{\nu j}_{1qln} \left[\frac{(q-1)!}{(q+1)!}\, e^{-2i\beta} B^{\nu}_{-1,q}\right] = \Phi^j_{1n}.$$

From the comparison of (59) with (40) and (41) we conclude that the relations (56), (57) are indeed valid. On the basis of (56), (57), the functions Π_j and Π^*_j, entering into the expressions for the Debye potentials (7), (8) of the scattered field, for $\alpha = 0$ may be written in the form of the expansions

$$\Pi_j = 2\,\frac{E_0}{ik}\,e^{i\beta}\cos(\varphi-\beta)\sum_{n=1}^{\infty} A^j_{1n}\psi_n(ka_j)h^{(1)}_n(kr_j)P^1_n(\cos\theta_j), \tag{60}$$

$$\Pi^*_j = 2\,\frac{E_0}{ik}\sqrt{\frac{\bar{\varepsilon}}{\mu}}\,e^{i\beta}\sin(\varphi-\beta)\sum_{n=1}^{\infty} B^j_{1n}\psi_n(ka_j)h^{(1)}_n(kr_j)P^1_n(\cos\theta_j). \tag{61}$$

In particular, if (53) holds, then approximately

$$A^j_{\pm1,n} = F^j_{\pm1,n} = -e^{iknr_{0j}}i^n\,\frac{2n+1}{n(n+1)}\,L^n_{\pm1,1}\,\frac{1}{\zeta^{(1)\prime}_n(ka_j)},$$

$$B^j_{\pm1,n} = \Phi^j_{\pm1,n} = -e^{iknr_{0j}}i^n\,\frac{2n+1}{n(n+1)}\,L^{*n}_{\pm1,1}\,\frac{1}{\zeta^{(1)}_n(ka_j)},$$

and then, for the scattered field, from (60), (61) we obtain

$$\Pi_j = -\frac{E_0e^{iknr_{0j}}}{ik}\cos(\varphi-\beta)\sum_{n=1}^{\infty} i^n\,\frac{2n+1}{n(n+1)}\,\frac{\psi'_n(ka_j)}{\zeta^{(1)\prime}_n(ka_j)}\times$$
$$\times h^{(1)}_n(kr_j)P^1_n(\cos\theta_j), \tag{62}$$

$$\Pi^*_j = -\frac{E_0e^{iknr_{0j}}}{ik}\sqrt{\frac{\bar{\varepsilon}}{\mu}}\,\sin(\varphi-\beta)\sum_{n=1}^{\infty} i^n\,\frac{2n+1}{n(n+1)}\,\frac{\psi_n(ka_j)}{\zeta^{(1)}_n(ka_j)}\times$$
$$\times h^{(1)}_n(kr_j)P^1_n(\cos\theta_j). \tag{63}$$

If (60), (61) are written in the form

$$\Pi_j = \cos(\varphi-\beta)\,\frac{dP^j}{d\theta_j}, \qquad \Pi^*_j = \sin(\varphi-\beta)\,\frac{dQ^j}{d\theta_j}, \tag{64}$$

where we have set

$$P^j = -\frac{2E_0}{ik^2r_j}\sum_{n=1}^{\infty} A^j_{1n}\psi_n(ka_j)\zeta^{(1)}_n(kr_j)P_n(\cos\theta_j), \tag{65}$$

and

$$Q^j = -\frac{i2E_0}{k^2 r_j}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=1}^{\infty} B_{1n}^j \psi_n(ka_j)\zeta_n^{(1)}(kr_j)P_n(\cos\theta_j), \tag{66}$$

then it is not difficult to observe that the functions (65), (66) are nothing other than a generalization of V. A. Fock' s potentials P, Q (see, for example, [24, 25]) to

case of several spheres under the condition of symmetry of the problem with respect to the axis $Oz(\alpha = 0)$. Indeed, if it is assumed that (53) is satisfied, then from the preceding results for P^j and Q^j we obtain the expressions

$$P^j = -\frac{E_0 e^{ik\mathbf{n}\mathbf{r}_{0j}}}{ik^2 r_j}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\frac{\psi_n'(ka_j)}{\zeta_n^{(1)\prime}(ka_j)}\zeta_n^{(1)}(kr_j)P_n(\cos\theta_j),$$

$$Q^j = \frac{E_0\sqrt{\dfrac{\varepsilon}{\mu}}\,e^{ik\mathbf{n}\mathbf{r}_{0j}}}{ik^2 r_j}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\frac{\psi_n(ka_j)}{\zeta_n^{(1)}(ka_j)}\zeta_n^{(1)}(kr_j)P_n(\cos\theta_j),$$

which coincide with the expressions for the potentials of V. A. Fock. Since, for the potentials of V. A. Fock, asymptotic representations corresponding to the case of large ka are well known, by means of which the series for P and Q are approximately summed, it becomes possible, when (53) and $\alpha = 0$ are satisfied, to obtain asymptotic expressions for the components of the scattered field for $ka \gg 1$ in the case of several strongly separated spheres.

We shall now find expressions for the fields and for some quantities characterizing them, considering, in order to simplify the reasoning, the special case of two spheres of equal radii.

3. The case of two spheres. For two spheres of equal radii $r_j = a,\ j = \pm 1$ (the coordinate systems are introduced as shown in Fig. 2; the spheres are assigned the indices $j = +1$ and $j = -1$), the Debye potentials of the scattered field will be determined by the expansions

$$\Pi^s = \sum_{j=\pm 1}\Pi_j, \qquad \Pi^{*s} = \sum_{j=\pm 1}\Pi_j^*, \tag{66'}$$

where Π_j and Π_j^* are given by formulas (9), (10), (39), while A_{mn}^j and B_{mn}^j are found from the systems

$$A_{mn}^j + \sum_{q=1}^{\infty}\gamma_{mnq}^{(-j,j)}A_{mq}^{-j} = F_{mn}^j, \tag{67}$$

Fig. 2

Figure 2: Fig. 2

$$B^j_{mn} + \sum_{q=1}^{\infty} \delta^{(-j,j)}_{mnq} B^{-j}_{mq} = \Phi^j_{mn} \tag{68}$$

($\gamma_{mnq}, \delta_{mnq}, F_{mn}$, and Φ_{mn} are the same as in (42), (43)). Here, in the right-hand parts of the systems, $\mathbf{nr}_{0j} = jl_0 \cos\alpha$, $j = \pm 1$. In the wave zone ($r_j \to \infty$, $r_j \gg l_0$), approximately

$$h_n^{(1)}(kr_j) = (-i)^{n+1} \frac{e^{ikr}}{kr} \, e^{-ikl_0 \cos\theta}, \qquad \theta_j \approx \theta.$$

Therefore, as follows from (6), for the scattered field $\mathbf{E}^s, \mathbf{H}^s$ in the approximation considered, $E_r^s = H_r^s = 0$, while $E_\theta^s, E_\varphi^s, H_\theta^s$, and H_φ^s decrease as $r_j \to \infty$ like r^{-1}. Consequently, as in the case of a single sphere, the scattered field in the wave zone is transverse in character. For the transverse field components E_θ^s, E_φ^s and H_θ^s, H_φ^s, from (6) one obtains the asymptotic expressions

$$E_\theta^s = \sqrt{\frac{\mu}{\varepsilon}} \, H_\varphi^s = \frac{iE_0 e^{ikr}}{kr} \, S(\theta, \varphi), \tag{69}$$

$$E_\varphi^s = -\sqrt{\frac{\mu}{\varepsilon}} \, H_\theta^s = \frac{iE_0 e^{ikr}}{kr} \, S^*(\theta, \varphi), \tag{70}$$

$$\begin{aligned} S(\theta, \varphi) = -\sum_{n,m} (-i)^n e^{im\varphi} \Bigg\{ & \psi_n'(ka) \times \\ & \times \frac{d}{d\theta} \left[P_n^m(\cos\theta) \sum_{j=\pm 1} A^j_{mn} e^{-ijkl_0 \cos\theta} \right] + \\ & + im \, \psi_n(ka) \frac{P_n^m(\cos\theta)}{\sin\theta} \sum_{j=\pm 1} B^j_{mn} e^{-ijkl_0 \cos\theta} \Bigg\}, \end{aligned} \tag{71}$$

Fig. 2

$$
\begin{aligned}
S^*(\theta,\varphi) = -\sum_{n,m}(-i)^n e^{im\varphi}\times \\
\times \left\{ im\,\psi_n'(ka)\frac{P_n^m(\cos\theta)}{\sin\theta}\times \right. \\
\times \sum_{j=\pm1} A_{mn}^j e^{-ijkl_0\cos\theta} - \psi_n(ka)\times \\
\left. \times \frac{d}{d\theta}\left[P_n^m(\cos\theta)\sum_{j=\pm1} B_{mn}^j e^{-ijkl_0\cos\theta}\right]\right\}.
\end{aligned}
\tag{72}
$$

From (69), (70) it is seen that in the far zone the electric and magnetic vectors of the scattered field are mutually perpendicular.

From the relation $\mathbf{j} = [\mathbf{nH}]$ for $r_j = a,\ j = \pm1$, we find for the components of the vector density of the surface currents induced on the surface of the j-th sphere that

$$
j_\theta^j = -H_\varphi, \qquad j_\varphi^j = H_\theta \quad (j=\pm1,\ r_j = a), \tag{73}
$$

where $\mathbf{H}$ denotes the vector of the total magnetic field. Therefore, putting in (6) $\Pi = \Pi^i + \Pi^s$ and $\Pi^* = \Pi^{*i} + \Pi^{*s}$, on the basis of (24)–(26), (66′) and (67), (68) we find that

$$
j_\theta^j = j_\theta^{ej} + j_\theta^{mj}, \qquad j_\varphi^j = j_\varphi^{ej} + j_\varphi^{mj}, \tag{74}
$$

where, if one uses the Wronskian determinant for the functions $\psi_n(ka)$ and $\zeta_n^{(1)}(ka)$, equal to

$$
\psi_n(x)\zeta_n^{(1)\prime}(x) - \psi_n'(x)\zeta_n^{(1)}(x) = i,
$$

$$
j_\theta^{ej} = -\frac{iE_0}{ka}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n,m} A_{mn}^j \frac{dP_n^m(\cos\theta_j)}{d\theta_j}e^{im\varphi},
$$

$$
j_\theta^{mj} = \frac{iE_0}{ka}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n,m} mB_{mn}^j \frac{P_n^m(\cos\theta_j)}{\sin\theta_j}e^{im\varphi}, \tag{75}
$$

$$
j_\varphi^{ej} = -\frac{E_0\sqrt{\frac{\varepsilon}{\mu}}}{ka}\sum_{n,m} mA_{mn}^j \frac{P_n^m(\cos\theta_j)}{\sin\theta_j}e^{im\varphi}, \tag{75}
$$

$$j_{\varphi}^{mj} = -\frac{E_0\sqrt{\frac{\bar{\varepsilon}}{\bar{\mu}}}}{ka} \sum_{n,m} B_{mn}^{j} \frac{dP_n^m(\cos\theta_j)}{d\theta_j} e^{im\varphi}.$$

In the case when $\alpha = 0$, instead of (71), (72) we obtain

$$S(\theta,\varphi) = -2e^{i\beta}\cos(\varphi-\beta)\sum_{n=1}^{\infty}(-i)^n \times$$

$$\times \left\{ \psi_n'(ka)\frac{d}{d\theta}\left[P_n(\cos\theta)\sum_{j=\pm1} A_{1n}^{j} e^{-ijkl_0\cos\Theta}\right] + \right.$$

$$\left. + i\psi_n(ka)\left[\sum_{j=\pm1} e^{-ijkl_0\cos\Theta} B_{1n}^{j}\right]\frac{dP_n(\cos\theta)}{\sin\theta\, d\theta}\right\}, \tag{76}$$

$$S^*(\theta,\varphi) = -2e^{i\beta}\sin(\varphi-\beta)\sum_{n=1}^{\infty}(-i)^n \times$$

$$\times \left\{ -\psi_n'(ka)\left[\sum_{j=\pm1} A_{1n}^{j} e^{-ijkl_0\cos\Theta}\right]\frac{dP_n(\cos\theta)}{\sin\theta\, d\theta} + \right.$$

$$\left. + i\psi_n(ka)\frac{d}{d\theta}\left[\frac{d}{d\theta}P_n(\cos\theta)\sum_{j=\pm1} B_{1n}^{j} e^{-ijkl_0\cos\Theta}\right]\right\}. \tag{77}$$

Using (71), (72), it is not difficult to find an expression for the quantity σ of the radar cross section of two spheres (the backscattering cross section), if one uses the formula

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|E_{\Theta}^{s}|^2}{|E_{\Theta}^{i}|^2}, \tag{78}$$

where the component E_{Θ}^{s} of the electric vector of the scattered field is taken in the direction toward the source. In particular, if, for example, $\alpha = \pi/2$ and $\beta = 0$, then for the component E_{Θ}^{i} of the field of the incident wave in the coordinates of the system $Oxyz$

$$E_{\Theta}^{i} = -E_0 e^{-ikx}\sin\theta,$$

and then from (78), on the basis of (69), (70), we find ($\varphi = 0,\ \theta = \pi/2$) that

$$\sigma = \frac{4\pi}{k^2}\left|\sum_{n,m}(-i)^n \left\{\psi_n'(ka)\times\right.\right.$$

$$\times \frac{d}{d\theta}\left[P_n^m(\cos\theta)\sum_{j=\pm1}A_{1n}^j e^{-ijkl_0\cos\Theta}\right]\Bigg|_{\Theta=\pi/2} +$$

$$\left.\left. + im\,\psi_n(ka)P_n^m(0)\sum_{j=\pm1}B_{1n}^j\right\}\right|^2, \tag{79}$$

where

$$P_n^m(0) = \begin{cases} 0, & \text{if } (n-m) \text{ is odd,} \\ \dfrac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n\left(\dfrac{n-m}{2}\right)!\left(\dfrac{n+m}{2}\right)!}, & \text{if } (n-m) \text{ is even,} \end{cases}$$

$$P_n^{m\,\prime}(0) = \begin{cases} 0, & \text{if } (n-m) \text{ is even,} \\ \dfrac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\dfrac{n-m-1}{2}\right)!\left(\dfrac{n+m+1}{2}\right)!}, & \text{if } (n-m) \text{ is odd.} \end{cases}$$

If, for example, $\alpha = 0$, $\beta = 0$, then for the field of the incident wave in the coordinates of the system $Oxyz$

$$E_\theta^i = E_0 e^{ikz}\cos\varphi\cos\theta,$$

and then $(\theta = \pi)$

$$\sigma = \frac{8\pi}{k^2}\left|\sum_{n=1}^{\infty}(-i)^n\left[\psi_n'(ka)\sum_{j=\pm1}e^{ijkl_0}A_{1n}^j\frac{d^2P_n(\cos\theta)}{d\theta^2} + i\psi_n(ka)\sum_{j=\pm1}e^{ijkl_0}B_{1n}^j\frac{dP_n(\cos\theta)}{\sin\theta\,d\theta}\right]\right|_{\theta=\pi}^2, \tag{80}$$

where

$$\lim_{\theta\to0}\frac{d^2P_n}{d\theta^2} = \lim_{\theta\to0}\frac{1}{\sin\theta}\frac{dP_n}{d\theta} = -\frac{n(n+1)}{2},$$

$$\lim_{\theta\to\pi}\frac{d^2P_n}{d\theta^2} = -\lim_{\theta\to\pi}\frac{1}{\sin\theta}\frac{dP_n}{d\theta} = -(-1)^n\frac{n(n+1)}{2}.$$

In exactly the same way, expressions can be obtained for the quantities considered above also in the case of N spheres forming a linear system.

In conclusion, let us note the following. To compute, for example, the coefficients A^j_{mn} $(j=\pm 1)$, system (67) may be written in the form

$$A^j_{mn} - \sum_{p=1}^{\infty}\left(\sum_{q=1}^{\infty}\gamma^{(-j,j)}_{mnq}\gamma^{(j,-j)}_{mqp}\right)A^j_{mnp} = F^j_{mn} - \sum_{q=1}^{\infty}\gamma^{(-j,j)}_{mnq}F^{-j}_{mq}. \tag{81}$$

In this form, system (81) has a very complicated structure, in consequence of which the calculation of the coefficients A^j_{mn} is a difficult matter. However, one can indicate another method for computing A^j_{mn} and B^j_{mn}, connected with the solution of systems of simpler structure than (81). For this purpose, we note that

$$\gamma^{(-j,j)}_{mnq} = (-1)^{n+q}\gamma^{(j,-j)}_{mnq}, \qquad \delta^{(-j,j)}_{mnq} = (-1)^{n+q}\delta^{(j,-j)}_{mnq}, \quad (j=\pm 1). \tag{82}$$

Indeed, from the fact that $P_\sigma(\cos\theta_{-j,j}) = 1$, if $\theta_{-j,j} = 0$, and $P_\sigma(\cos\theta_{-j,j}) = (-1)^\sigma$, if $\theta_{-j,j} = \pi$, where σ has the same parity as $n+q$ (the Clebsch-Gordan coefficients entering expression (29) for b_σ are ···

are different from zero only for those n, q, and σ whose sum $n+q+\sigma$ is even, we have

$$Q_{mnmq}(-j,j) = (-1)^{n+q}Q_{mnmq}(j,-j). \tag{83}$$

From (83) and (42), (43), (82) follows. On the basis of (82), from (67) we obtain

$$A^j_{mn} + \sum_{q=1}^{\infty}\gamma^{(-j,j)}_{mnq}A^{-j}_{mq} = F^j_{mn},$$

$$(-1)^n A^{-j}_{mn} + \sum_{q=1}^{\infty}(-1)^q\gamma^{(j,-j)}_{mnq}A^j_{mq} = (-1)^n F^{-j}_{mn},$$

whence, after termwise addition and subtraction of the equalities, we find that

$$z_{mn} + \sum_{q=1}^{\infty}(-1)^q\gamma^{(-j,j)}_{mnq}z_{mq} = f_{mn},$$

$$u_{mn} - \sum_{q=1}^{\infty}(-1)^q\gamma^{(-j,j)}_{mnq}u_{mq} = \varphi_{mn}, \tag{84}$$

where $z_{mn} = A^j_{mn} + (-1)^n A^{-j}_{mn}$, $u_{mn} = A^j_{mn} - (-1)^n A^{-j}_{mn}$, $f_{mn} = F^j_{mn} + (-1)^n F^{-j}_{mn}$, and $\varphi_{mn} = F^j_{mn} - (-1)^n F^{-j}_{mn}$. In an analogous way, from (68) we obtain

$$c_{mn} + \sum_{q=1}^{\infty} (-1)^q \delta^{(-j,j)}_{mnq} c_{mq} = \tilde{f}_{mn},$$

$$d_{mn} - \sum_{q=1}^{\infty} (-1)^q \delta^{(-j,j)}_{mnq} d_{mq} = \tilde{\varphi}_{mn}, \qquad (85)$$

where $c_{mn} = B^j_{mn} + (-1)^n B^{-j}_{mn}$, $d_{mn} = B^j_{mn} - (-1)^n B^{-j}_{mn}$, $\tilde{f}_{mn} = \Phi^j_{mn} + (-1)^n \Phi^{-j}_{mn}$, $\tilde{\varphi}_{mn} = \Phi^j_{mn} - (-1)^n \Phi^{-j}_{mn}$. Systems (84), (85) have a much simpler structure than systems of the form (81).

References

1. Trinks W. Ann. der Phys., **22**, 561–590, 1935.
2. Mevel J. Annales de Physique, **5**, No. 3–4, 265–320, 1960.
3. Germogenova O. A. Izv. AN SSSR, ser. geofizicheskaya, No. 4, 648–653, 1963.
4. Ivanov E. A. Izv. vuzov, radiofizika, **6**, No. 6, 1155–1166, 1963.
5. Kumagai N., Angelakos D. J. Institute of Engineering Res. Report, N 116, University of California. Berkeley, 1960.
6. Twersky V. J. math Phys., **3**, 83–91, 1962.
7. Agelakas D. J., Kumagai N. IEEE, Trans. Antennas and Propagation, **AP-12**, No. 1, 105–109, 1964.
8. Moe K. A., Angelakos D. J. ERL Tech. Rept. Series 60, No. 365, Univ of California, Berkeley, June, 1961.
9. Burke J. E., Twersky V. Radio Science, **68D**, N 4, Sec. D., J. Res. Nat. Bureau of Standards, 500–510, 1964.
10. Mentzer J. R. *Diffraction and Scattering of Waves.* Sov. Radio Publishing House, Moscow, 1958.

11. Khenl H., Maué A., Westpfahl K. *Theory of Diffraction.* Mir Publishing House, Moscow, 1964.

12. Debyl P. Ann. Physik (4), **30**, 57, 1909.

13. Born M. *Optics.* ONTI, Kharkov–Kiev, 1937.

14. Gel' fand I. M., Minlos R. A., Shapiro Z. Ya. *Representation of the Rotation Group and the Lorentz Group.* Fizmatgiz, Moscow, 1958.

15. Hobson E. V. *The Theory of Spherical and Ellipsoidal Functions.* IL, 1952.

16. Ivanov E. A., *DAN BSSR*, **4**, No. 1, 3–6, 1960.

17. Meixner J., Schäfke F. *Matieusche Funktionen und Sphäroid funktionen.* Springer–Verlag. Berlin, 1954.

18. Friedman B., Russek J. *Quart. Appl. Math.*, **12**, No. 1, 13–23, 1954.

19. Sack R. A. *J. Math. Phys.*, **5**, 2, 252–259, 1964.

20. Lyubarskii G. Ya. *Group Theory and Its Application in Physics.* Gostekhizdat, 1957.

21. Yutsis A. P., Levinson I. B., Vanagas V. V. *The Mathematical Apparatus of the Theory of Angular Momentum.* Publishing House for Political and Scientific Literature of the Lithuanian SSR, Vilnius, 1960.

22. Kantorovich L. V., Krylov V. I. *Approximate Methods of Higher Analysis.* Fizmatgiz, Moscow, 1962.

23. Gradshteyn I. S., Ryzhik I. M. *Tables of Integrals, Sums, Series, and Products.* Fizmatgiz, Moscow, 1962.

24. Fock V. A. *ZhETF*, **19**, issue 10, 916–929, 1949.

25. Collection "Diffraction of Electromagnetic Waves by Certain Bodies of Revolution." Publishing House "Sov. radio," Moscow, 1957, pp. 57–125.

26. Vilenkin N. Ya. *Special Functions and the Theory of Group Representations.* Nauka, Moscow, 1965.

Received by the editors
January 3, 1966

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Note: Figure translations are in progress. See original paper for figures.