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ON THE DECOMPOSITION OF THE DIRECT PRODUCT OF REPRESENTATIONS OF THE DISCRETE UNITARY SERIES D_λ^+ OF THE LORENTZ GROUP $\mathcal{L}_3$

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In this note we shall prove the following formula for the decomposition of the direct product of representations of the discrete unitary series D_λ^+ of the Lorentz group $\mathcal{L}_3$:

$$D_{\lambda_1}^+ \otimes D_{\lambda_2}^+ \otimes \cdots \otimes D_{\lambda_n}^+ = \sum_{k=0}^{\infty} \oplus C_k^{n+k-2} D^+_{\lambda_1+\lambda_2+\cdots+\lambda_n+k}, \tag{I}$$

which is very useful in describing the n-dimensional oscillator from the point of view of its dynamical symmetry group ([1]). In (I) the binomial coefficient C_k^{n+k-2} denotes the multiplicity of the irreducible representation $D^+_{\lambda_1+\cdots+\lambda_n+k}$ in the decomposition.

First of all, we shall carry out explicitly the decomposition of the direct product $D_{\lambda_1}^+ \otimes D_{\lambda_2}^+$ into irreducible components (we do not restrict ourselves to one-valued or two-valued representations ([2]), but take λ in D_λ^+ to be any positive real number). In doing so we use the basis method ([3]).*

It should be noted that the formulas obtained below are relatively simple in comparison with the corresponding formulas for the unitary representations of the continuous series C_q^λ, considered in Pukanszky' s work ([5]).

Let $\{f_{n_1}^{(\lambda_1)}\}_{n_1=0}^\infty$ and $\{f_{n_2}^{(\lambda_2)}\}_{n_2=0}^\infty$ be the canonical bases of the irreducible representations $D_{\lambda_1}^+$ and $D_{\lambda_2}^+$ in the Hilbert spaces $H_{\lambda_1}^+$ and $H_{\lambda_2}^+$, forming orthonormal systems with respect to the scalar products in $H_{\lambda_1}^+$ and $H_{\lambda_2}^+$: $(f_{n_1}^{(\lambda_1)}, f_{n_1'}^{(\lambda_1)}) = \delta_{n_1 n_1'}$, $(f_{n_2}^{(\lambda_2)}, f_{n_2'}^{(\lambda_2)}) = \delta_{n_2, n_2'}$. The functions $f_n^{(\lambda)}$ satisfy the equations (H_0, H_1, H_2 are the generators of infinitesimal transformations of the group $\mathcal{L}_3$**, $H_\pm = H_1 \pm iH_2$):

$$H_0 f_n^{(\lambda)} = (n+\lambda) f_n^{(\lambda)}, \qquad H_+ f_n^{(\lambda)} = a_{n+1}^{(\lambda)} f_{n+1}^{(\lambda)}, \qquad H_- f_n^{(\lambda)} = a_n^{(\lambda)} f_{n-1}^{(\lambda)}, \quad (1)$$

where

$$a_n^{(\lambda)} = \sqrt{n(n+2\lambda-1)}, \quad n = 0, 1, 2, \ldots \ (^2)\,.$$

The basis of the direct-product representation $D_{\lambda_1}^+ \otimes D_{\lambda_2}^+$ is the system of functions $\{f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)}\}$, forming an orthonormal system in $H_{\lambda_1}^+ \otimes H_{\lambda_2}^+$ with respect to the scalar product

$$(f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)},\, f_{n_1'}^{(\lambda_1)} f_{n_2'}^{(\lambda_2)}) = (f_{n_1}^{(\lambda_1)}, f_{n_1'}^{(\lambda_1)})(f_{n_2}^{(\lambda_2)}, f_{n_2'}^{(\lambda_2)}) = \delta_{n_1 n_1'} \delta_{n_2 n_2'}.$$

The operators H_a $(a = 0, 1, 2)$ act on the functions $f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)}$ according to the formulas

$$H_a (f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)}) = (H_a f_{n_1}^{(\lambda_1)}) f_{n_2}^{(\lambda_2)} + f_{n_1}^{(\lambda_1)} (H_a f_{n_2}^{(\lambda_2)}). \quad (2)$$

(Obviously, the operators H_a are Hermitian and give a representation of the infinitesimal group $\mathcal{L}_3$ in $H_{\lambda_1}^+ \otimes H_{\lambda_2}^+$.) In particular,

$$H_0 f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)} = (n_1 + n_2 + \lambda_1 + \lambda_2) f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)}, \quad (3)$$

* In (^{4}) this method was applied in the decomposition of the direct product of representations of the Lorentz group $\mathcal{L}_4$.

** The operators H_0, H_1, H_2 satisfy the commutation relations $[H_1, H_2] = -iH_0$, $[H_2, H_0] = iH_1$, $[H_0, H_1] = iH_2$.

so that to the eigenvalue $n + \lambda_1 + \lambda_2$ of the operator H_0 there correspond $n+1$ functions: $f_0^{(\lambda_1)} f_n^{(\lambda_2)}, f_1^{(\lambda_1)} f_{n-1}^{(\lambda_2)}, \ldots, f_n^{(\lambda_1)} f_0^{(\lambda_2)}$, where $n = n_1 + n_2 \geqslant 0$. The latter condition means that in the decomposition of the representation $D_{\lambda_1}^+ \otimes D_{\lambda_2}^+$ only representations of the series D_λ^+ $(\lambda > 0)$ will occur. It is easy to find these representations. Obviously, $H_- f_0^{(\lambda_1)} f_0^{(\lambda_2)} = 0$, $H_0 f_0^{(\lambda_1)} f_0^{(\lambda_2)} = (\lambda_1 + \lambda_2) f_0^{(\lambda_1)} f_0^{(\lambda_2)}$. Hence $f_0^{(\lambda_1)} f_0^{(\lambda_2)}$ is a basis element with the least eigenvalue $\lambda_1 + \lambda_2$ of the representation $D_{\lambda_1+\lambda_2}^+$. With the aid of the operators $H_+, H_+^2, H_+^3, \ldots$, from $f_0^{(\lambda_1)} f_0^{(\lambda_2)}$ one can obtain all the other basis elements of the representation $D_{\lambda_1+\lambda_2}^+$. In this process the elements $f_0^{(\lambda_1)} f_1^{(\lambda_2)}$ and $f_1^{(\lambda_1)} f_0^{(\lambda_2)}$ will enter into $D_{\lambda_1+\lambda_2}^+$ in a certain linear combination. Their other combination will be a basis element with the least eigenvalue $\lambda_1 + \lambda_2 + 1$ of the representation $D_{\lambda_1+\lambda_2+1}^+$. From it, with the aid of the operators $H_+, H_+^2, H_+^3, \ldots$, one can obtain all the other basis elements of the representation $D^+_{\lambda_1+\lambda_2+1}$. Repeating this procedure, we construct bases of all the other representations $D^+_{\lambda_1+\lambda_2+2}$, $D^+_{\lambda_1+\lambda_2+3}$, ..., occurring in the decomposition of the representation $D^+_{\lambda_1} \otimes D^+_{\lambda_2}$. Thus we obtain the decomposition

$$D^+_{\lambda_1} \otimes D^+_{\lambda_2} = \sum_{k=0}^{\infty} D^+_{\lambda_1+\lambda_2+k}. \tag{4}$$

Let us now show that on the right there is an orthogonal sum. First of all, from the construction of the basis elements of the representation D^+_λ ($\lambda = \lambda_1 + \lambda_2 + k$) it follows that $f_n^{(\lambda)} \sim H_+^n f_0^{(\lambda)}$, while $f_{n'}^{(\lambda')} \sim H_+^{n'} f_0^{(\lambda')}$. Then (in H^+_λ, $\lambda = \lambda_1 + \lambda_2 + k$, the same scalar product is defined as in $H^+_{\lambda_1} \otimes H^+_{\lambda_2}$)

$$(f_n^{(\lambda)}, f_{n'}^{(\lambda')}) \sim (H_+^n f_0^{(\lambda)}, H_+^{n'} f_0^{(\lambda')}). \tag{5}$$

By virtue of the Hermiticity of the operators H_α we have $(H_+ f, g) = (f, H_- g)$, so that for $n \neq n'$ the scalar product (5) vanishes because $H_-^{n-n'} f_0^{(\lambda')} = 0$ $(n > n')$ or $H_-^{n'-n} f_0^{(\lambda)} = 0$ $(n' > n)$. For $n = n'$ we have $(f_n^{(\lambda)}, f_{n'}^{(\lambda')}) \sim (f_0^{(\lambda)}, f_0^{(\lambda')})$. If $\lambda \neq \lambda'$, then the last expression, as is not hard to verify*, vanishes. Thus in (4) there is an orthogonal sum. Therefore, if $f, g \in H^+_{\lambda_1} \otimes H^+_{\lambda_2}$, then

$$(f, g) = \sum_\lambda (f^{(\lambda)}, g^{(\lambda)}), \tag{6}$$

where $f^{(\lambda)}, g^{(\lambda)} \in H^+_\lambda$, $\lambda = \lambda_1 + \lambda_2 + k$ ($f^{(\lambda)}$ is the orthogonal projection of f onto H^+_λ). From equality (6) it follows that there must exist an isometric mapping of the space $H^+_{\lambda_1} \otimes H^+_{\lambda_2}$ onto the space $\sum_{k=0}^{\infty} \oplus H^+_{\lambda_1+\lambda_2+k}$. This means that the representations $D^+_{\lambda_1} \otimes D^+_{\lambda_2}$ and $\sum_{k=0}^{\infty} D^+_{\lambda_1+\lambda_2+k}$ are isometrically equivalent.

Let us now find the explicit form of this mapping, i.e., express the basis elements $f_n^{(\lambda)} \in H^+_\lambda$ $(\lambda = \lambda_1 + \lambda_2 + k)$ through the elements $f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)} \in H^+_{\lambda_1} \otimes H^+_{\lambda_2}$. From (3) it follows that

$$f_n^{(\lambda)} = \sum_{n_1=0}^{k+n} C^{\lambda n}_{\lambda_1 n_1; \lambda_2 n_2} f_{n_1}^{(\lambda_1)} f_{n_2}^{(\lambda_2)}, \qquad n_1 + \lambda_1 + n_2 + \lambda_2 = n + \lambda. \tag{7}$$

We need to find the explicit form of the Clebsch–Gordan coefficients $C^{\lambda n}_{\lambda_1 n_1; \lambda_2 n_2}$. For this purpose we temporarily pass to new bases $\xi_n^{(\lambda)} = \gamma_n^{(\lambda)} f_n^{(\lambda)}$

* For this it is necessary to turn to formula (7).

with $\gamma_n^{(\lambda)} = \sqrt{n!\Gamma(n + 2\lambda)}$. Then we obtain the simple formula $H_+ \xi_n^{(\lambda)} = \xi_{n+1}^{(\lambda)}$, and, consequently,

$$\xi_n^{(\lambda)} = H_+^n \xi_0^{(\lambda)}. \tag{8}$$

Here $H_- \xi_n^{(\lambda)} = \beta_n^{(\lambda)} \xi_{n-1}^{(\lambda)}$, where $\beta_n^{(\lambda)} = n(n+2\lambda-1)$. For the element $\xi_0^{(\lambda)}$ we have the expansion

$$\xi_0^{(\lambda)} = \sum_{p=0}^{k} a_p \xi_p^{(\lambda_1)} \xi_{k-p}^{(\lambda_2)}.$$

The coefficients a_p are found from the condition $H_- \xi_0^{(\lambda)} = 0$, which leads to the equality $a_p \beta_p^{(\lambda_1)} + a_{p-1} \beta_{k-p+1}^{(\lambda_2)} = 0$. Hence we obtain

$$a_p = a_0 (-1)^p \left[p! \Gamma(p+2\lambda_1)(k-p)! \Gamma(k-p+2\lambda_2)\right]^{-1},$$

where a_0 is as yet undetermined. From (8) and property (2) it now follows that

$$\xi_n^{(\lambda)} = \sum_{m=0}^{n} \sum_{p=0}^{k} C_m^n a_p \xi_{p+m}^{(\lambda_1)} \xi_{k+n-p-m}^{(\lambda_2)} \tag{9}$$

(C_m^n is the number of combinations of n taken m at a time). Passing to the old bases and comparing formulas (9) and (7), we find the expression for the Clebsch-Gordan coefficients*

$$C_{\lambda_1 n_1; \lambda_2 n_2}^{\lambda n} = a_0 \left(\frac{n!\, n_1!\, n_2!\, \Gamma(n_1+2\lambda_1)\Gamma(n_2+2\lambda_2)}{\Gamma(n+2\lambda)} \right)^{1/2} \times$$

$$\times \sum_{p=\max(0, n_1-n)}^{\min(n_1, \lambda-\lambda_1-\lambda_2)} (-1)^p \left[p! (\lambda-\lambda_1-\lambda_2-p)! (n_1-p)! (n_1+p-n_1)! \times\right.$$

$$\left. \times \Gamma(p+2\lambda_1)\Gamma(\lambda-\lambda_1+\lambda_2-p)\right]^{-1}, \tag{10}$$

where $\lambda = \lambda_1 + \lambda_2 + k$ $(k = 0, 1, 2, ...)$, and $n_2 = n + k - n_1$.

The coefficient a_0 in (10) is found from the normalization condition $(f_0^{(\lambda)}, f_0^{(\lambda)}) = 1$ and is equal to

$$a_0 = [(2\lambda-1)(\lambda-\lambda_1-\lambda_2)! \times$$

$$\times \Gamma(\lambda+\lambda_1-\lambda_2)\Gamma(\lambda+\lambda_2-\lambda_1)\Gamma(\lambda+\lambda_1+\lambda_2-1)]^{1/2}. \tag{11}$$

Starting from formula (4), it is easy to prove formula (1). It is obtained by repeated application of formula (4). In doing so one must use the following property of the binomial coefficients:

$$\sum_{k'=0}^{k} C_{k'}^{n+k'} = C_{k}^{n+1+k}.$$

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* Analogous formulas for the Clebsch–Gordan coefficients can be obtained for the discrete unitary series D_λ^- ($\lambda > 0$), and also for finite-dimensional representations of the group $\mathcal{L}_3$.

Note: Figure translations are in progress. See original paper for figures.