

TRANSFORMATION OPERATORS AND THE SCATTERING FUNCTION IN THE CASE OF A HIGHLY SINGULAR POTENTIAL

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Abstract

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MATHEMATICS

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TRANSFORMATION OPERATORS AND THE SCATTERING FUNCTION IN THE CASE OF A HIGHLY SINGULAR POTENTIAL

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1. We consider the radial scattering equation

$$y'' + \{k^2 - V(x)\}y = 0, \quad 0 < x < \infty, \quad (1)$$

with a repulsive potential of the form $V(x) = W(x) + U(x)$, where $W(x) \rightarrow +\infty$ sufficiently rapidly as $x \rightarrow 0$, and $U(x)$ is a relatively small perturbation. It is also assumed that for any $a > 0$

$$\int_a^\infty x|V(x)| dx < \infty. \quad (2)$$

Denote by $f(x, k)$ the solution of equation (1) determined by the condition

$$\lim_{x \rightarrow \infty} f(x, k)e^{-ikx} = 1. \quad (3)$$

It is known that, in solving the inverse problem of scattering theory and in a number of other questions, the application of transformation operators of the form

$$f(x, k) = e^{ikx} + \int_x^\infty K(x, t)e^{ikt} dt, \quad (4)$$

which were introduced by B. Ya. Levin ⁽¹⁾ and studied in detail in the works of Z. S. Agranovich and V. A. Marchenko ^(2,3), has proved very fruitful. In this setting, the singularity of the potential as $x \rightarrow 0$ was in any case allowed to be no higher than Cx^{-2} . We shall assume that the potential (more precisely, $W(x)$) has as $x \rightarrow 0$ a singularity of higher order,* than x^{-2} , that $W(x)$ is three times continuously differentiable for $x > 0$, and that as $x \rightarrow 0$

$$W'(x) = O(W^{3/2-\delta}(x)), \quad 0 < \delta < \frac{1}{2}; \quad (5)$$

$$W''(x)/W'(x) = O(W'(x)/W(x)), \quad W'''/W'' = O(W''/W'). \quad (6)$$

Let, moreover, $W(x)$, together with its three derivatives, be monotone for $0 < x < 1$, let $W(x) \equiv 0$ for $x > 1$, and let the function $U(x)$ be measurable and

$$\int_0^{1/2} |U(x)| \cdot |W'(x)|^{-1/3} dx < \infty. \quad (7)$$

Conditions (5), (6) are satisfied, in particular, for functions of the form Cx^{-p} for any $p > 2$, $C > 0$ (with its own δ for each p), and also for $e^{1/x}$, $\exp\{e^{1/x}\}$, etc.

* This follows from the remaining conditions.

We denote

$$\gamma(x) = 2^{-1/2}W^{-1/4}(x) \exp \left\{ \int_x^1 W^{1/2}(t) dt \right\}, \quad (8)$$

$$\gamma_0(x) = 2^{-1/2}W^{-1/4}(x) \exp \left\{ - \int_x^1 W^{1/2}(t) dt \right\}. \quad (9)$$

It is easy to see that $\gamma(x) \rightarrow +\infty$, $\gamma_0(x) \rightarrow 0$ as $x \rightarrow 0$.

Theorem 1. Let $K(x, t)$ be the kernel of the transformation operator (4) for equation (1) with potential $V(x) = W(x) + U(x)$, satisfying the conditions listed above. Then, as $x \rightarrow 0$, there exists the limit

$$\lim_{x \rightarrow 0} K(x, t)/\gamma(x) = K(t), \quad 0 < t < \infty, \quad (10)$$

where $K(t) \not\equiv 0$ is an infinitely differentiable function, bounded together with each of its derivatives $K^{(n)}(t)$ in the metrics $C[0, \infty)$ and $L^p[0, \infty)$ for any $p \geq 1$, and

$$K(0) = K^{(n)}(0) = 0, \quad n = 1, 2, \dots \quad (11)$$

Convergence to the limit (10) takes place in $C[0, \infty)$ and in $L^p[0, \infty)$ for $p \geq 1$.

The main stages of the proof are outlined below in Sec. 3.

2. Equation (1), under conditions close to ours, was studied in the works of Limić ⁽⁴⁾. In particular, he constructed a solution $\varphi(x, k)$, regular as $x \rightarrow 0$, for which

$$\lim_{x \rightarrow 0} \varphi(x, k)/\gamma_0(x) = 1, \quad (12)$$

defined the Jost function

$$f(k) = W\{f(x, k), \varphi(x, k)\} = f\varphi' - f'\varphi, \quad (13)$$

and also established the existence of the scattering function

$$S(k) = \lim_{x \rightarrow 0} f(x, -k)/f(x, k) = f(-k)/f(k). \quad (14)$$

These results of Limić are also valid in our case.

Applying similar arguments, one can also establish the following theorem.

Theorem 2. For any solution $y(x, k)$ of equation (1) with a potential of the kind considered, satisfying condition (5) and the condition

$$\int_0^{1/2} |U(x)| W^{-1/2}(x) dx < \infty, \quad (15)$$

there exist equal finite limits

$$\lim_{x \rightarrow 0} y(x, k)/\gamma(x) = \lim_{x \rightarrow 0} y'(x, k)/\gamma'(x) = y(k), \quad (16)$$

where $y(k) = 0$ only in the case when $y(x, k) = C\varphi(x, k)$, and then

$$\lim_{x \rightarrow 0} y'(x, k)/\gamma'_0(x) = \lim_{x \rightarrow 0} y(x, k)/\gamma_0(x). \quad (17)$$

We note that condition (15) is weaker than (7), in view of (5).

Corollary 1. For any solution $y(x, k)$

$$\lim_{x \rightarrow 0} W^{-1/2}(x)y'(x, k)/y(x, k) = \mp 1, \quad (18)$$

where the sign + occurs only if $y(x, k) = C\varphi(x, k)$.

Corollary 2. The Jost function (13) is equal to

$$f(k) = \lim_{x \rightarrow 0} f(x, k)/\gamma(x). \quad (19)$$

3. Denote by $X(k)$ the root of the equation

$$W(X) = k^2, \quad (20)$$

$$\alpha(k) = -k + \int_{X(k)}^1 \{k^2 - W(t)\}^{1/2} dt, \quad k > 0. \quad (21)$$

Theorem 3. Under the conditions of Theorem 1, as $k \rightarrow +\infty$,

$$f(k) \sim \gamma^{-1}(X) \exp \left\{ -i \left[\frac{\pi}{4} + \alpha(k) \right] - k^2 \int_0^X \frac{dt}{\sqrt{W(t) - k^2} + \sqrt{W(t)}} \right\}, \quad (22)$$

and for some $C > 0$, $C_1 > 0$,

$$|f(k)| \leq C_1 \exp\{-C|k|^{2\delta}\}, \quad -\infty < k < \infty. \quad (23)$$

The **proof** is based on the study of the integral equation for $f(x, k)$ by means of Langer's method ⁽⁵⁾, which is generalized to the case of potentials of arbitrary rate of growth as $x \rightarrow 0$.

Let

$$\Phi_1(x, k) = f(x, k)/\gamma(x), \quad \Phi_2(x, k) = \Phi_1(x, k) - e^{ikx}/\gamma(x),$$

$$\Phi_3(x, k) = \Phi_2(x, k) - \frac{1}{\gamma(x)} \int_x^\infty \frac{\sin k(t-x)}{k} V(t) e^{ikt} dt.$$

Lemma 1. The convergence $\lim_{x \rightarrow 0} \Phi_j(x, k) = f(k)$ takes place in the metric $C(-\infty, \infty)$ for all $j = 1, 2, 3$, and also in $L^p(-\infty, \infty)$ for $\Phi_2(x, k)$ when $p > 1$ and for $\Phi_3(x, k)$ when $p > 1/2$.

Lemma 2. For any $n \geq 1$,

$$\sup_{-\infty < k < \infty} \left| k^n \frac{d}{dk} f(k) \right| < \infty. \quad (24)$$

Theorem 3, Lemmas 1, 2, and the theorem on convergent sequences of Hermitian-positive functions ⁽⁶⁾ lead to a complete proof of Theorem 1, if the following is taken into account.

Remark. If $V(x) \geq 0$, then $f(k)$ is Hermitian-positive. If $V_2(x) \geq |V_1(x)|$, then $K_2(x, t) \geq |K_1(x, t)|$, where K_1 and K_2 are the kernels of the corresponding transformation operators.

4. **Theorem 4.** Under the conditions of item 1, the Jost function $f(k)$ admits the integral representation

$$f(k) = \int_0^{\infty} K(t)e^{ikt} dt, \quad (25)$$

where $K(t)$ is defined by (10) and possesses all the properties indicated in Theorem 1.

The following theorem follows from Theorem 3 and makes it possible, from the asymptotics of the scattering function $S(k)$, to find the asymptotics of the potential as $x \rightarrow 0$.

Theorem 5. Under the conditions of item 1, when $k \rightarrow +\infty$,

$$S(k) = \exp\{i[2\alpha(k) + \pi/2 + o(1)]\}, \quad (26)$$

where $\alpha(k)$ is defined by (21) and, as is known, $S(-k) = S^{-1}(k)$. If, moreover, there exists, for some $p > 2$,

$$\lim_{x \rightarrow 0} W(x)x^p = C \neq 0,$$

then, as $k \rightarrow +\infty$,

$$S(k) = i \exp\{-2ikX(k)[A_p + o(1)]\}, \quad (27)$$

where

$$A_p = 1 + \frac{1}{p} \int_0^1 (1 - \sqrt{1-t}) t^{-1-1/p} dt. \quad (28)$$

Formulas (27), (28) are also valid when, for every $\varepsilon > 0$, as $x \rightarrow 0$ $W(x)x^{p+\varepsilon} \rightarrow 0$, $W(x)x^{p-\varepsilon} \rightarrow \infty$ monotonically in some neighborhood of $x = 0$. Finally, if $[\ln W(x)]/|\ln x| \rightarrow \infty$ monotonically as $x \rightarrow 0$, then in (27) one should put $A_p = A_\infty = 1$.

For an analytic potential $V(z)$, admitting for $\operatorname{Re} z > 0$ the estimate $|V(z)| < C|z|^{-4}$, an asymptotic formula similar to (26) was obtained in ⁴. (There, however, the term $\pi/2$ was omitted.)

5. Let us consider the Cauchy problem for the equation

$$F''_{xx}(x, y) - V(x)F(x, y) = F''_{yy}(x, y) \quad (29)$$

with initial data $F(1, y)$, $F'_x(1, y)$, $-\infty < y < \infty$, which are generalized functions ⁷ over the basic space K of finite, infinitely differentiable functions. For each $x > 0$ the solution $F(x, y)$ is a generalized function over K .

Theorem 6. If $V(x)$ in (29) satisfies the conditions of item 1, then as $x \rightarrow 0$ $F(x, y)/\gamma(x) \rightarrow F(y)$ (in the sense of convergence of generalized functions), where $F(y)$ turns out to be an ordinary, infinitely differentiable function.

There exist initial data for which $F(y) \neq 0$, but if*

$$F(x, y) = (R(k), \varphi(x, k)e^{iky}), \quad R(k) \in Z', \quad \varphi(x, k)$$

is defined by (12), then $F(y) \equiv 0$.

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* From (13) and (23) it follows easily that for any $x > 0$, $\varphi(x, k) \in Z$. For the definition of Z, Z' , see ⁷.

Note: Figure translations are in progress. See original paper for figures.

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