ON MULTIFREQUENCY OSCILLATIONS AND ROTATIONS

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Abstract

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§ 1. **Statement of the problem.** In this article we study the system

$$dx_i/dt = F_i(t, x_1, x_2, \dots, x_n) + \varepsilon f_i(t, x_1, x_2, \dots, x_n; \varepsilon) \qquad (i = 1, 2, \dots, n), \quad (1)$$

where $t \in (-\infty, \infty)$ is an independent variable; $\varepsilon \in [0, \varepsilon_0]$ is a small parameter; $\{x_i\} \in E^n$; the functions F_i and f_i satisfy the following conditions:

1) F_i are periodic in $t, x_1, x_2, \dots, x_p$ $(p \leq n)$ with periods $T, T_1, T_2, \dots, T_p$ for all real values of these arguments, while f_i, uniformly with respect to $\varepsilon, x_{p+1}, \dots, x_n$, are almost periodic (in the sense of (1′)) in $t, x_1, x_2, \dots, x_p$ with frequency bases $\{\omega_m\}, \{\omega_{m_1}\}, \{\omega_{m_2}\}, \dots, \{\omega_{m_p}\}$ (2), respectively, where $\{\omega_m\}, \{\omega_{m_1}\}, \{\omega_{m_2}\}, \dots, \{\omega_{m_p}\}$ are, generally speaking, countable sequences.

2) F_i possess second, and f_i first, partial derivatives satisfying Lipschitz conditions in x_i, ε with constants independent of t in the region $G \in E^n$, unbounded along the coordinate axes $x_1, x_2, \dots, x_p$, and $\varepsilon \in [0, \varepsilon_0]$, and are continuous in t.

Along with the perturbed system (1), consider the unperturbed system

$$\frac{dx_i^0}{dt} = F_i(t, x_1^0, x_2^0, \dots, x_n^0) \qquad (2)$$

and suppose that in the region G it admits an isolated rotational-oscillatory solution of the form (see (3))

$$x_i^0 = v_i(t) = \delta_i \frac{T_i}{2\pi} \omega t + u_i(t), \qquad (3)$$

where $\omega = 2\pi/T$; $u_i(t + T) = u_i(t)$; $\delta_i = 1$ for $i \leq p$ and $\delta_i = 0$ for $i > p$.

We shall say that the solution (3) is isolated if the characteristic equation for the system in variations

$$dz_i/dt = p_{i1}z_1 + p_{i2}z_2 + \dots + p_{in}z_n, \tag{4}$$

where $p_{ik}(t) = \partial F_i/\partial x_k|_{x_k=v_k} = p_{ik}(t+T)$, has no roots with moduli equal to 1 (see $(4')$). Under these conditions the system (4) has no periodic $(4, 5)$, nor almost-periodic solutions.

We pose the problem of constructing and investigating an almost-rotational-oscillatory solution of the system (1) (definition below), with any degree of accuracy in ε for all $t \in (-\infty, \infty)$, lying in the region G and passing into the generating solution (3) when $\varepsilon = 0$.

Systems of type (1), when the generating solution is periodic, were considered in (4). Systems with rotations and oscillations have already been studied with the aid of various modifications of the averaging method (1), as a rule on a bounded time interval $\sim 1/\varepsilon$ or $\sim 1/\sqrt{\varepsilon}$ (3, 6–11). In the present work stationary motions on the unbounded time interval $t \in (-\infty, \infty)$ are studied.

§ 2. **Main results.** Under the assumptions made concerning F_i, f_i, and the system of variational equations (4), the following assertion holds:

Theorem. If the generating system (2) has an isolated solution (3), then the perturbed system (1), for sufficiently small $|\varepsilon|$, admits a unique solution belonging to the region G and reducing to the generating one (3) when $\varepsilon = 0$, and, when $\varepsilon \neq 0$, having the form

$$x_i = v_i(t) + \overline{y}_i(t, \varepsilon) \qquad (i = 1, 2, \dots, n),$$

where $\overline{y}_i$ is almost-periodic in t, with basis

$$\{\omega, \{\omega_m\}, \{\Omega_{m_1}\}, \{\Omega_{m_2}\}, \dots, \{\Omega_{m_p}\}\} \qquad \left(\Omega_{m_k} = \frac{T_k}{2\pi}\omega\omega_{m_k}\right), \tag{5}$$

and satisfies, with respect to ε, Lipschitz conditions with constants independent of t. This perturbed solution is asymptotically stable for $t \geq t_0$, for sufficiently small $|\varepsilon|$, if the characteristic exponents of the variational system (4) have negative real parts, and is unstable if at least one of them is positive.

§ 3. **Construction of successive approximations.** In (1) put $x_i = v_i + y_i$. The system (1) takes the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_{ik}y_k + V_i(t, y_1, \dots, y_n) + \varepsilon(f_i)_0 + \varepsilon W_i(t, y_1, \dots, y_n; \varepsilon), \tag{6}$$

where

$$V_i = F_i(t, x_1, x_2, \dots, x_n) - (F_i)_0 - \sum_{k=1}^{n} p_{ik} y_k, \tag{7}$$

$$W = f_i(t, x_1, x_2, \dots, x_n; \varepsilon) - (f_i)_0. \tag{8}$$

For sufficiently small values of $|y_i|$, $|y_i'|$, $|y_i''|$, $|\varepsilon|$, $|\varepsilon'|$, $|\varepsilon''|$, these functions satisfy the conditions

$$|V_i(t, y)| < A \sum_{k=1}^{n} y_k^2 \qquad (y \equiv \{y_1, y_2, \dots, y_n\}), \tag{9}$$

$$|V_i(t, y') - V_i(t, y'')| < B \sum_{k=1}^{n} (|y_k'| + |y_k''|) \sum_{l=1}^{n} |y_l' - y_l''|;$$

$$|W_i(t, y, \varepsilon)| < C \left(\sum_{k=1}^{n} |y_k| + |\varepsilon| \right);$$

$$|W_i(t, y', \varepsilon') - W_i(t, y'', \varepsilon'')| < C \left(\sum_{k=1}^{n} |y_k' - y_k''| + |\varepsilon' - \varepsilon''| \right). \tag{10}$$

We seek the solution of system (6) by successive approximations, taking $y_i^{(0)} = 0$ as the zeroth approximation. The first is found from the system

$$\frac{dy_i^{(1)}}{dt} = \sum_{k=1}^{n} p_{ik} y_k^{(1)} + \varepsilon (f_i)_0, \tag{11}$$

and an arbitrary l-th one ($l \geq 2$)

$$\frac{dy_i^{(l)}}{dt} = \sum_{k=1}^{n} p_{ik} y_k^{(l)} + \varepsilon (f_i)_0 + V_i(t, y^{(l-1)}) + \varepsilon W_i(t, y^{(l-1)}; \varepsilon). \tag{12}$$

The existence and uniqueness of an almost-periodic solution of system (11) is ensured by Lyapunov' s theorem on reducibility, by means of a nonsingular periodic matrix, of system (11) to a system with constant coefficients ([4,5]), and also by the well-known theorem of Neugebauer–Bohr ([2,4]).

Thus, one can formally construct any approximation to the solution of system (6). If these successive approximations converge to some almost-periodic function, then their limit may represent the desired solution.

§ 4. **Justification of the approximations.** Let us show, first, that any approximation is bounded. For the first approximation this is true on the basis of the theorems mentioned:

$$\left|y_i^{(1)}\right| < |\varepsilon| P\Phi \qquad (i = 1, 2, \dots, n),$$

where Φ is the upper limit of the functions $|(f_i)_0|$, and the constant P depends only on p_{ik}. Next suppose that the inequality is valid for $l = 0, 1, 2, \dots, m-1$:

$$\left|y_i^{(l)}\right| < |\varepsilon| P\Phi < |\varepsilon| N, \tag{13}$$

where N is some constant. We shall show that the last inequality (13) is also valid for $l = m$. Indeed, taking into account (9), (10), (12), (13), we obtain

$$\left|y_i^{(m)}\right| < |\varepsilon| P\Phi + \varepsilon^2 nAN^2P + \varepsilon^2 C(nN+1)P < |\varepsilon| N \qquad (m = 0, 1, 2, \dots), \tag{14}$$

provided

$$P\Phi + |\varepsilon| nAN^2P + |\varepsilon| C(nN+1)P < N,$$

which is evidently true for sufficiently small values of $|\varepsilon|$.

From the obtained inequality two consequences follow:

1) $x_i = v_i + y_i \in G$, if $v_i \in G$ and $|\varepsilon|$ is small;

2) $x_i|_{\varepsilon=0} = v_i(t)$.

Secondly, we prove the uniform convergence of $y_i^{(l)}$ to certain almost-periodic functions $\overline{y_i}(t)$. For this purpose consider the differences $y_i^{(l+1)} - y_i^{(l)}$. They satisfy the system

$$\frac{d\left(y_i^{(l+1)} - y_i^{(l)}\right)}{dt} = \sum_{k=1}^{n} p_{ik}\left(y_k^{(l+1)} - y_k^{l}\right) + \left\{V_i\left(t, y^{(l)}\right) - V_i\left(t, y^{(l-1)}\right)\right\} +$$

$$+\varepsilon\left\{W_i\left(t, y^{(l)}; \varepsilon\right) - W_i\left(t, y^{(l-1)}; \varepsilon\right)\right\}.$$

On the basis of (9), (10), (13), (14),

$$\left|y_i^{(l+1)} - y_i^{(l)}\right| < |\varepsilon| a_l nP(2nBN + C),$$

where a_l is the upper limit of the quantities $\left|y_i^{(l)} - y_i^{(l-1)}\right|$. Setting

$$a_{l+1} = |\varepsilon| a_l nP(2nBN + C),$$

we see that $y_i^{(l)}$ converges uniformly. Since all $y_i^{(l)}$ are almost-periodic, the limit, according to a theorem of the theory of almost-periodic functions ([2]), will be an almost-periodic function with basis (5).

Thirdly, $\overline{y_i}(t)$ satisfy system (6). To prove this fact, denote by z_i the unique almost-periodic solution of the system

$$\frac{dz_i}{dt} = \sum_{k=1}^{n} p_{ik} z_k + V_i(t, \overline{y}) + \varepsilon (f_i)_0 + \varepsilon W_i(t, \overline{y}; \varepsilon).$$

Analogously to the preceding, we find

$$\left|z_i - y_i^{(l)}\right| < |\varepsilon| nP(2nBN + C) \left|\overline{y_i} - y_i^{(l-1)}\right|_{\max}.$$

Since

$$\left|\overline{y_i} - y_i^{(l)}\right|_{\max} \to 0 \quad \text{as } l \to \infty,$$

passing to the limit, we obtain $z_i \equiv \overline{y_i}$, and hence it follows that $\overline{y_i}(t)$ is a solution of (6).

Let us finally prove that, for sufficiently small $|\varepsilon|$, the functions $\bar{y}_i(t)$ determine in the domain $\bar{G}$ ($\dim \bar{G} \sim |\varepsilon|$) the unique almost-periodic solution of system (6). Suppose that $\bar{\bar{y}}_i(t)$ is also an almost-periodic solution. Then one can write

$$|\bar{\bar{y}}_i - \bar{y}_i| < |\varepsilon nP(2nBN + C)^M| \, |\bar{\bar{y}}_i - \bar{y}_i|_{\max},$$

where M is an arbitrarily large integer. Hence $\bar{\bar{y}}_i(t) \equiv \bar{y}_i(t)$. This completes the proof of the theorem.

Remark. The theorem, as is not difficult to see from (7)–(10) and the estimates obtained, remains valid if $\partial F_i / \partial x_k$ and f_i satisfy Lipschitz conditions with constants independent of t in x_k, ε in an analogous domain G and $\varepsilon \in [0, \varepsilon_0]$.

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Note: Figure translations are in progress. See original paper for figures.