EMBEDDING THEOREMS FOR A CLASS OF FUNCTIONS WITH MIXED NORM

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EMBEDDING THEOREMS FOR A CLASS OF FUNCTIONS WITH MIXED NORM

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In this note we consider a class of differentiable functions defined on the n-dimensional space R_n, whose derivatives of a certain order have finite mixed norm ([1]), while the higher derivatives satisfy a multiple Hölder condition in this same norm. For functions of this class, which we denote by $S_p^{(\mathbf{r})}H(R_n)$, where $\mathbf{r} = (r_1, \dots, r_n)$, $\mathbf{p} = (p_1, \dots, p_n)$ (for the definition of the class $S_p^{(\mathbf{r})}H(R_n)$, see ([2])), Ya. S. Bugrov ([2]) proved a theorem on representation in the form of a series of entire functions satisfying certain conditions.

Theorem (main). In order that a function $f(\mathbf{x})$ belong to the class

$$S_p^{(\mathbf{r})}H(R_n), \qquad \mathbf{r} = (r_1, \dots, r_n), \quad r_i > 0 \quad (i = 1, \dots, n),$$
$$\mathbf{p} = (p_1, \dots, p_n), \qquad 1 \le p_i \le \infty \quad (i = 1, \dots, n),$$

it is necessary and sufficient that it be representable in the form

$$f(\mathbf{x}) = \sum_{e \subseteq e_n} \sum_{k^e \ge 0} Q^e_{k^e},$$

where the outer sum, of a finite number of terms (series), is extended over all possible subsets $e \subseteq e_n$, including the empty set. The inner sum is extended over all integer nonnegative vectors $\mathbf{k}^e = (k_1^e, \dots, k_i^e)$, $k_j^e \ge 0$. The functions $Q_{k^e}(\mathbf{x})$ are entire of degree $2^{k_j^e}$ in x_j, $j \in e$ (thus, of degree 1 in x_j, $j \in e_n - e$), and satisfy the inequalities

$$\|Q_{k^e}\|_p \le M \cdot 2^{-(\mathbf{k}, \mathbf{r}^e)},$$

where M is a constant.

Using the conditions of the main theorem, we have proved several embedding theorems for the classes $S_p^{(\mathbf{r})}H(R_n)$, generalizing results of S. M. Nikol' skii ([3]).

Theorem 1. Let the function $f(\mathbf{x})$ belong to the class $S_p^{(\mathbf{r})}H(R_n)$, and let $0 < \rho \leq \mathbf{r}$, where $\mathbf{r} = (r_1, \dots, r_n)$, $r_j > 0$, $\rho = (\rho_1, \dots, \rho_n)$, $\rho_j > 0$, $\mathbf{p} = (p_1, \dots, p_n)$. Then

$$f(\mathbf{x}) \in S_p^{(\rho)}H(R_n),$$
$$\|f\|_{S_p^{(\rho)}H(R_n)} \leq c\|f\|_{S_p^{(\mathbf{r})}H(R_n)}.$$

Theorem 2. Let the function $f(\mathbf{x})$ belong simultaneously to the classes

$$S_p^{(\mathbf{r}^1)}H(R_n), \dots, S_p^{(\mathbf{r}^N)}H(R_n).$$

and

$$\sum_1^N \lambda_\nu \leqslant 1, \qquad \lambda_\nu \geqslant 0 \quad (\nu = 1, \dots, N), \qquad \mathbf{r} = \sum_1^N \lambda_\nu \mathbf{r}^\nu.$$

Then

$$f(x) \in S_p^{(\mathbf{r})}H(R_n);$$

$$\|f\|_{S_p^{(\mathbf{r})}H(R_n)} \leqslant c \prod_{\nu=1}^N \|f\|^{\lambda_\nu}_{S_p^{(\mathbf{r}^\nu)}H(R_n)}, \qquad \text{for } \sum_1^N \lambda_\nu = 1;$$

$$\|f\|_{S_p^{(\mathbf{r})}H(R_n)} \leqslant c \prod_{\nu=1}^N \|f\|^{\lambda_\nu/\alpha}_{S_p^{(\mathbf{r}^\nu)}H(R_n)}, \qquad \text{for } \sum_1^N \lambda_\nu = \alpha < 1;$$

$$\|f\|_{S_p^{(\mathbf{r})}H(R_n)} \leqslant c \sum_{\nu=1}^N \|f\|_{S_p^{(\mathbf{r}^\nu)}H(R_n)}, \qquad \text{for } \sum_1^N \lambda_\nu \leqslant 1.$$

Theorem 3. Let the function $f(x)$ belong to the class

$$S_p^{(\mathbf{r})}H(R_n), \qquad \mathbf{r} = (r_1, \dots, r_n) > 0 \quad (e_r = e_n),$$

$$\mathbf{p} = (p_1, \dots, p_n), \qquad p_1 \geqslant p_2 \geqslant \dots \geqslant p_n \geqslant 1, \qquad \mathbf{q} = (q_1, \dots, q_n),$$

$$p_k < q_k \leqslant \infty \qquad (k = 1, n).$$

Then $f(x) \in S_q^{(\vec{\rho})}H(R_n)$, where $\vec{\rho} = (\rho_1, \dots, \rho_n)$,

$$\rho_j = r_j - \left(\frac{1}{p_j} - \frac{1}{q_j}\right) > 0 \qquad (j = 1, \dots, n),$$

$$\|f\|_{S_q^{(\vec{\rho})} H(R_n)} \leqslant c\|f\|_{S_p^{(\mathbf{r})} H(R_n)}.$$

Theorem 4. Let the function $f(x)$ belong to the class

$$S_p^{(\mathbf{r})} H(R_n), \qquad \mathbf{r} = (r_1, \dots, r_n) > 0 \quad (e_r = e_n);$$

$x^0_{m+1}, \dots, x^0_n$ are fixed coordinates of the point $\mathbf{x} = (x_1, \dots, x_m, x_{m+1}, \dots, x_n)$, $1 \leqslant m < n$.

Suppose, moreover, that for some nonnegative numbers $\lambda_{m+1}, \dots, \lambda_n$, forming the system $(\vec{\lambda})$, the inequalities

$$r_j - \lambda_j - \frac{1}{p_j} > 0 \qquad (j = m + 1, \dots, n)$$

are satisfied. Then

$$f^{(\lambda)} = \frac{\partial^{\lambda_{m+1}+\dots+\lambda_n} f(x_1, \dots, x_m, x^0_{m+1}, \dots, x^0_n)}{\partial^{\lambda_{m+1}} x_{m+1} \cdots \partial^{\lambda_n} x_n} = \psi(\mathbf{y}, \mathbf{z}_0)$$

as a function of $(x_1, \dots, x_m)$ belongs to the class $S_p^{(\vec{\rho})} H(R_m)$, where

$$\vec{\rho} = (r_1, \dots, r_m) = \mathbf{r}^{e_m}, \qquad \mathbf{y} = (x_1, \dots, x_m), \qquad \mathbf{z}_0 = (x^0_{m+1}, \dots, x^0_n),$$

and

$$\|\psi\|_{S_p^{(\vec{\rho})} H(R_m)} \leqslant c\|\psi\|_{S_p^{(\mathbf{r})} H(R_n)},$$

$$\|\psi(\mathbf{y}, \mathbf{z}) - \psi(\mathbf{y}, \mathbf{z}_0)\|_{(p_1, \dots, p_m)} \to 0 \quad \text{as } |\mathbf{z} - \mathbf{z}_0| \to 0,$$

where $\mathbf{z} = (x_{m+1}, \dots, x_n)$, $\psi(\mathbf{y}, \mathbf{z}) = f^{(\vec{\lambda})}(\mathbf{x})$.

If

$$\sum_{m+1}^{n} \lambda_j = 0,$$

then $f^{(\vec{\lambda})}$ is the function f itself.

Theorem 5. Let a system of functions be given

$$\varphi_{(\vec{\lambda})}(x_1, \dots, x_m), \qquad (\vec{\lambda}) = (\lambda_{m+1}, \dots, \lambda_n),$$

where λ_i are nonnegative integers ($i = m+1, \dots, n$), belonging to the class $S_{\vec{\rho}}^{(p)} H(R_m)$, $\vec{\rho} = (r_1, \dots, r_m)$, $r_i > 0$ ($i = 1, \dots, m$).

Then one can construct a function $f(x_1, \dots, x_n)$ of n variables having the following properties:

a) $f \in S_p^{(\mathbf{r})} H(R_n)$, $\mathbf{r} = (r_1, \dots, r_n)$, where for r_j ($j \geq m+1$) one may take arbitrarily large natural numbers, and, moreover,

$$\|f\|_{S_p^{(\mathbf{r})} H(R_n)} \leq c \sum_{(\vec{\lambda})} \|\varphi_{(\vec{\lambda})}\|_{S_p^{(\vec{\rho})} H(R_m)};$$

b)

$$\frac{\partial^{\lambda_{m+1}+\dots+\lambda_n} f(x_1, \dots, x_m, 0, \dots, 0)}{\partial^{\lambda_{m+1}} x_{m+1} \cdots \partial^{\lambda_n} x_n} = \varphi_{(\vec{\lambda})}(x_1, \dots, x_m).$$

The last equality is understood in the sense of convergence in norm (1).

If

$$\sum_{m+1}^{n} \lambda_j = 0,$$

then the left-hand side in condition b) is the function f itself.

Remark. In the proof of the theorems, an essential role was played by inequalities (11), (16), obtained by I. I. Ibragimov[4].

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