ON THE DIVERGENCE OF FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.512

MATHEMATICS

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ON THE DIVERGENCE OF FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 5 II 1966)

1. Denote by B^* the class of uniformly almost periodic (a.p.) functions $f(x)$ whose Fourier exponents have a single limit point λ^*. If $\lambda^* \neq \infty$, then without loss of generality one may assume that $\lambda^* = 0$, $M\{f(x)\} = 0$, since otherwise it would suffice to consider the function

$$F(x) = f(x)e^{-i\lambda^* x} - M\{f(x)e^{-i\lambda^* x}\}.$$

Let $\{\lambda_k\}$ $(k = 1, 2, \dots;\ \lambda_k > 0)$ be the monotone sequence formed by the absolute values of the Fourier exponents of the function $f(x) \in B^*$; then the Fourier series of this function is naturally written in the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \quad (\lambda_0 = 0,\ \lambda_{-k} = -\lambda_k,\ |A_k| + |A_{-k}| > 0 \text{ for } k > 0). \quad (1)$$

We shall assign $f(x) \in B^*$ to the class B_0^* if $\lambda_k \downarrow 0$ and $M\{f(x)\} = 0$; we shall assign $f(x) \in B^*$ to the class B_∞^* if $\lambda_k \uparrow \infty$. We shall say that $f(x) \in B_0^*$ belongs to the class $B_0^{*[p]}$ $(p = 1, 2, \dots)$ if there exist functions $f_0(x), f_1(x), \dots, f_p(x)$ possessing the following properties: $f_0(x) = f(x)$, $f'_{m+1}(x) = f_m(x)$ $(m = 0, 1, 2, \dots, p-1)$, $f_m(x) \in B_0^*$ $(m = 0, 1, 2, \dots, p)$. Put $B_0^* = B_0^{*[0]}$; the inclusions

$$B_0^{*[0]} \supset B_0^{*[1]} \supset B_0^{*[2]} \supset \cdots$$

are obvious. It is easy to see that a function $f(x) \in B_0^{*[m]}$ belongs to the class $B_0^{*[m+1]}$ if and only if the indefinite integral of the function

$$f_m(x) \sim \sum_{k=-\infty}^{\infty} \frac{A_k}{(i\lambda_k)^m} e^{i\lambda_k x}$$

is bounded.

We shall assign $f(x) \in B_\infty^*$ to the class $B_\infty^{*(p)}$ if on the whole number axis there exists for $f(x)$ a uniformly continuous derivative of order p. Put $B_\infty^* = B_\infty^{*(0)}$; the inclusions

$$B_\infty^{*(0)} \supset B_\infty^{*(1)} \supset B_\infty^{*(2)} \supset \cdots$$

are obvious.

2. The following criteria for convergence of Fourier series of uniform a.p. functions of the class B^* hold.

Theorem A. Let $f(x) \in B^*_\infty$, $f(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$. If

$$\lambda_n^{-\alpha} \ln \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} = o(1), \tag{2}$$

then the series (1) converges uniformly to $f(x)$ on the whole real axis.

Theorem B. Let $f(x) \in B^*_0$ and let there exist a constant C such that

$$\left| \int_0^u f(x+w)\, dw \right| < C|u|^{1-\alpha} \quad (0 < \alpha \le 1). \tag{3}$$

If

$$\lambda_n^{\alpha} \ln \frac{\lambda_n + \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = o(1), \tag{2'}$$

then the series (1) converges uniformly to $f(x)$ for all real x.

The first of these theorems is due to S. Bochner ([1,2]), the second to B. M. Levitan ([2,3]).

In this note it is asserted (Theorems 1 and 2) that Theorem A and Theorem B for $\alpha < 1$ are final in the sense defined below; new final criteria (Theorems 3 and 4) are given for the uniform convergence of Fourier series of functions of the classes $B^{*(p)}_\infty$, $B^{*[p]}_0$.

3. **Theorem 1.** There exists a function $f(x) \in B^*_\infty$, $f(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ such that

$$\lambda_n^{-\alpha} \ln \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} = O(1) \tag{4}$$

and the Fourier series (1) of this function diverges at the point $x = 0$.

Theorem 2. There exists a function $f(x) \in B^*_0$, satisfying condition (3) for $0 < \alpha < 1$, for which

$$\lambda_n^{\alpha} \ln \frac{\lambda_n + \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = O(1) \tag{4'}$$

and the Fourier series (1) diverges at every point of the real axis.

The proofs of Theorems 1 and 2 are based on the following constructions. Let $\{\Lambda_n\}$ $(n = 1, 2, ...)$ be a monotone sequence of positive numbers; let the numbers

$\varepsilon_n > 0$ $(n = 1, 2, ...)$ be chosen so that the intervals $(\Lambda_n - \varepsilon_n, \Lambda_n + \varepsilon_n)$ $(n = 1, 2, ...)$ do not intersect, and let $\{N_n\}$ $(n = 1, 2, ...)$ be an increasing sequence of natural numbers. We introduce into consideration trigonometric polynomials analogous to Fejér polynomials ([4]):

$$Q(x, n) = P(x, n) - \widetilde{P}(x, n),$$

where

$$P(x, n) = \sum_{k=1}^{N_n} \frac{1}{k} \cos\left(\Lambda_n - k\frac{\varepsilon_n}{N_n}\right) x, \qquad \widetilde{P}(x, n) = \sum_{k=1}^{N_n} \frac{1}{k} \cos\left(\Lambda_n + k\frac{\varepsilon_n}{N_n}\right) x.$$

Lemma 1. For all values of x and n,

$$|Q(x, n)| < C, \tag{5}$$

where C is an absolute constant.

Lemma 2. If $\Lambda_n \uparrow \infty$, then for all n

$$P(0, n) > \ln N_n. \tag{6}$$

If $\Lambda_n \downarrow 0$, then for each x there exists $M > 0$ such that for $n > M$

$$P(x, n) > \tfrac{1}{2} \ln N_n. \tag{7}$$

Put

$$f(x) = \sum_{n=1}^{\infty} \frac{Q(x, n)}{2^{n\alpha}} \qquad (0 < \alpha < 1). \tag{8}$$

In view of (5), $f(x) \in B^*$, and the series (8) is the Fourier series of the function $f(x)$; it is easy to see that this series can be written in the form (1) by first introducing in (8) a single enumeration of the frequencies (where the frequencies of the polynomials $Q(x, n)$ are arranged in decreasing order when $\Lambda_n \downarrow 0$, and in increasing order when $\Lambda_n \uparrow \infty$).

Lemma 3. For $\Lambda_n = 2^n$, $\varepsilon_n = 2^{n-2}$, and any natural N_n,

$$f(x) \in \operatorname{Lip} \alpha.$$

Lemma 4. For $\Lambda_n = 2^{-n}$, $\varepsilon_n = 2^{-n-2}$, and any natural N_n, estimate (3) holds, where $C = C(\alpha)$ is a constant depending only on α.

Proof of Theorem 1. The conditions of the theorem are satisfied by the function $f(x)$, defined by the series (8), for $\Lambda_n = 2^n$, $\varepsilon_n = 2^{n-2}$, $N_n = [2^{n\alpha}]$. Indeed, by Lemma 3, $f(x) \in \operatorname{Lip}\alpha$; for the sequence of its Fourier exponents

$$\left\{\Lambda_n - k\frac{\varepsilon_n}{N_n}\right\} \quad (k = N_n, N_n - 1, \dots, 1)$$

$$\left\{\Lambda_n + k\frac{\varepsilon_n}{N_n}\right\} \quad (k = 1, 2, \dots, N_n)$$

$$(n = 1, 2, \dots)$$

condition (4) is fulfilled; by (6),

$$\lim_{n\to\infty} P(0, n)/2^{n\alpha} \neq 0,$$

therefore Cauchy' s convergence criterion is not fulfilled, and the Fourier series (1) of the function $f(x)$ diverges at the point $x = 0$.

Proof of Theorem 2. The conditions of the theorem are satisfied by the function $f(x)$, defined by the series (8), for $\Lambda_n = 2^{-n}$, $\varepsilon_n = 2^{-n-2}$, $N_n = [2^{n\alpha}]$. Indeed, by Lemma 4, $f(x)$ satisfies condition (3); for the sequence of its Fourier exponents

$$\left\{\Lambda_n + k\frac{\varepsilon_n}{N_n}\right\} \quad (k = N_n, N_n - 1, \dots, 1)$$

$$\left\{\Lambda_n - k\frac{\varepsilon_n}{N_n}\right\} \quad (k = 1, 2, \dots, N_n)$$

$$(n = 1, 2, \dots)$$

condition (4′) is fulfilled, and, by (7), the Fourier series (1) of the function $f(x)$ diverges at every point of the real axis.

Thus, conditions (2) and (2′) in Theorems A and B (for $\alpha < 1$) for the classes B^*_∞ and B^*_0 in general cannot be weakened without imposing additional conditions on the structural properties of the function $f(x) \in B^*$.

For $\alpha = 1$, Theorem B admits a refinement, for, by (3), $f(x) \in B_0^{*[1]}$ and, by Theorems 2 and 5 of paper [6], uniform convergence of the series (1) will be ensured if in condition (2′) one replaces $o(1)$ by $O(1)$.

4. There are generalizations [5,6] of Theorems A and B that make it possible to use more fully both the behavior of the Fourier exponents of the function $f(x) \in B^*$ and its structural properties. We give here two new convergence criteria, which are in fact contained in the results of papers [5,6] and which contain Theorems A and B.

A direct consequence of Theorem 6 of paper [5] is

Theorem 3. Let $f(x) \in B_\infty^{*(p)}$ $(p = 0, 1, 2, ...)$, $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$. If

$$\lambda_n^{-(p+\alpha)} \ln \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} = o(1), \tag{9}$$

then the series (1) converges uniformly to $f(x)$ on the entire real axis.

From the corollary of Theorem 2 and Theorem 5 of paper [6] it follows that

Theorem 4. Let $f(x) \in B_0^{*[p]}$ $(p = 0, 1, 2, ...)$, and suppose there exists a constant C such that

$$\left| \int_0^u f_p(x+u)\, du \right| < C|u|^{1-\alpha} \quad (0 < \alpha \leq 1).$$

If

$$\lambda_n^{p+\alpha} \ln \frac{\lambda_n + \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = o(1), \tag{9'}$$

then the series (1) converges uniformly to $f(x)$ for all real x.

Theorems 3 and 4 are final in the same sense as Theorems A and B: in (9) and (9′) (for $\alpha < 1$), generally speaking, one cannot replace $o(1)$ by $O(1)$. The proof of this assertion is carried out in the same way as ···

scheme, as in the proofs of Theorems 1 and 2, but requires more complicated preliminary constructions.

Let us note in conclusion that, in contrast to the periodic case, no improvement of the structural properties of the function $f(x) \in B^*$ by itself ensures uniform convergence of the Fourier series (1). Thus, for example, the function $f(x)$ defined by the series (8) with $\Lambda_n = n$, $\varepsilon_n = 1/3$, and $N_n = [2^{n^\alpha}]$, is infinitely differentiable on the entire real axis and has a Fourier series (1) diverging at the point $x = 0$.

The author expresses gratitude to P. L. Ulyanov for posing the problem and for his attention to the work.

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Received
5 II 1966

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Note: Figure translations are in progress. See original paper for figures.