ON THE THEORY OF GEODESIC MAPPINGS OF RIEMANNIAN SPACES

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Abstract

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ON THE THEORY OF GEODESIC MAPPINGS OF RIEMANNIAN SPACES

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The paper considers and in principle solves the problem of determining all Riemannian spaces admitting a nontrivial geodesic mapping onto a given V_n. In addition, an estimate is given for the degree of arbitrariness in the solution of this problem, and necessary and sufficient conditions of algebraic character are obtained, in invariant form, for V_n' s admitting a nontrivial geodesic mapping. The investigation is carried out in the class of analytic functions, locally, without restriction on the signature of the spaces.

1. A Riemannian space V_n with metric tensor g_{ij} $(i, j = 1, 2, \dots, n)$ admits a nontrivial geodesic mapping if and only if there exists a solution $\breve{g}_{ij} (= \breve{g}_{ji})$ of the system of equations

$$\breve{g}_{ij,k} = \lambda_i g_{kj} + \lambda_j g_{ki}, \qquad (A_1)$$

where $\lambda_i \neq 0$ is a gradient vector, and the comma denotes the sign of covariant differentiation in V_n. In this case the metric tensor $\bar{g}_{ij}$ of the corresponding space $\bar{V}_n$ is determined from the relations

$$\bar{g}_{ij} = e^{2\psi} \breve{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \qquad \psi_{,i} = -\lambda_\alpha \breve{g}^{\alpha\beta} g_{\beta i}, \qquad (1)$$

where $\breve{g}^{ij}$ are the elements of the inverse matrix for $\|\breve{g}_{ij}\|$ (1). Nonvanishing of the determinant of the solution $\breve{g}_{ij}$ of equations (A_1) can always be achieved by adding to it a term of the form $c g_{ij}$ $(c = \text{const})$.

From the integrability conditions of system (A_1) we find

$$\breve{g}_{\alpha j} R^{\alpha}{}_{ikl} + \breve{g}_{\alpha k} R^{\alpha}{}_{ilj} + \breve{g}_{\alpha l} R^{\alpha}{}_{ijk} = 0, \qquad (2)$$

$$\breve{g}_{\alpha i} R^{\alpha}{}_{j} - \breve{g}_{\alpha j} R^{\alpha}{}_{i} = 0, \qquad (3)$$

$$n\lambda_{i,j} = \mu g_{ij} + \breve{g}_{\alpha i} R^{\alpha}{}_{j} - \breve{g}_{\alpha\beta} R^{\alpha}{}_{ji}{}^{\beta}, \tag{A_2}$$

as a consequence of which they reduce to the form

$$\breve{g}_{\alpha_3} T^{\alpha_3}{}_{ijkl} = 0, \tag{B_1}$$

$$T^{\alpha\beta}{}_{ijkl} = n\left(\delta^{\alpha}_{i} R^{\beta}{}_{jkl} - \delta^{\alpha}_{k} R^{\beta}{}_{l(ij)}\right) - g_{jk}\left(\delta^{\alpha}_{i} R^{\beta}{}_{l} - R^{\alpha}{}_{il}{}^{\beta}\right) + g_{il}\left(\delta^{\alpha}_{k} R^{\beta}{}_{j} - R^{\alpha}{}_{kj}{}^{\beta}\right).$$

In turn, from the integrability conditions of equations A_2 it follows that

$$(n-1)\mu_{,k} = 2(n+1)\lambda_{\alpha} R^{\alpha}{}_{k} - \breve{g}_{\alpha\beta}\left(R^{\alpha\beta}{}_{,k} - 2R^{\alpha}{}_{k,}{}^{\beta}\right), \tag{A_3}$$

and they are represented in the form

$$\begin{aligned}(n+3)\lambda_{\alpha} R^{\alpha}{}_{ilk} = {} & \frac{1}{n-1} g_{il}\left[(n+3)\lambda_{\alpha} R^{\alpha}{}_{k} - \breve{g}_{\alpha_3}\left(R^{\alpha_3}{}_{,k} - 2R^{\alpha}{}_{k,}{}^{\beta}\right)\right] \\ & - \frac{1}{n-1} g_{ik}\left[(n+3)\lambda_{\alpha} R^{\alpha}{}_{l} - \breve{g}_{\alpha_3}\left(R^{\alpha_3}{}_{,l} - 2R^{\alpha}{}_{l,}{}^{\beta}\right)\right] \\ & + \breve{g}_{\alpha i} R^{\alpha}{}_{[l,k]} + \breve{g}_{\alpha_3} R_{lki}{}^{\beta,\alpha}.\end{aligned} \tag{B_2}$$

Finally, the integrability conditions of the equations A_3 (taking into account A_1 and A_2) give the relations

$$\frac{n+1}{n}\breve{g}_{\alpha\beta}\left(R^{\alpha}{}_{k\gamma\cdot}{}^{\beta} R^{\gamma}{}_{l} - R^{\alpha}{}_{\cdot l\gamma}{}^{\beta} R^{\gamma}{}_{k}\right) + (n+3)\lambda_{\alpha} R^{\alpha}{}_{[k,l]} - \breve{g}_{\alpha_3}\left(\frac{1}{2} R^{\alpha\beta}{}_{,[kl]} - R^{\alpha}{}_{k,\cdot l}{}^{\beta} + R^{\alpha}{}_{\cdot l,\cdot k}{}^{\beta}\right). \tag{B_3}$$

Thus, we obtain a system of differential equations A_{123} of the first order with respect to g_{ij}, λ_i, μ, solved with respect to their derivatives, with coefficients from V_n and integrability conditions B_{123}, and hence also the theorem:

Theorem 1. *In order that V_n admit a nontrivial geodesic mapping, it is necessary and sufficient that the system of equations A have a nontrivial solution $\breve{g}_{ij} \neq c g_{ij}$, $\lambda_i \neq 0$.*

Since the general solution of the system A is determined from it (in the form of a Taylor series) uniquely up to the initial conditions $\overset{0}{g}_{ij}$, $\overset{0}{\lambda}_{i}$, $\overset{0}{\mu}$, which can be chosen arbitrarily only in the case of its complete integrability, it follows from B that

Theorem 2. *The general space $\bar{V}_n$ admitting a nontrivial geodesic mapping onto a given V_n is determined by this space uniquely up to $r \leq (n+1)(n+2)/2$ arbitrary parameters. This maximum is attained if and only if V_n is a space of constant curvature.*

The relations B, the first B^1, the second B^2, and so on, their continuations, are linear algebraic equations with respect to $\breve{g}_{ij}$, λ_i, and μ, with coefficients from V_n. Since the solution of the system A corresponding to initial values satisfying the indicated relations satisfies them identically, we obtain the theorem:

Theorem 3. *In order that V_n admit a nontrivial geodesic mapping, it is necessary and sufficient that the system of equations B, B^1, $B^2, \ldots$ have a nontrivial solution with respect to $\breve{g}_{ij}$ and λ_i.*

Analysis of the result of covariant differentiation of the conditions B_1 shows that when $\breve{g}_{\alpha\beta} T^{\alpha\beta}_{ij\,kl,m} = 0$ by virtue of B_1, $\lambda_i = 0$. This occurs for V_n satisfying the relations

$$T^{(\alpha 3)}_{ij\quad kl,m} = Q^{pqrs}_{ijklm} T^{(\alpha 3)}_{pq\quad rs}. \tag{4}$$

Consequently, we have the theorem:

Theorem 4. *Riemannian spaces satisfying conditions (4) do not admit a nontrivial geodesic mapping.*

2. Consider the relations (2), writing them in the form

$$\breve{g}_{\alpha 3}\, \tilde{T}^{\alpha 3}_{ij\,kl} = 0,$$

$$\tilde{T}^{\alpha 3}_{ij\,kl} = \delta^{(\alpha}_j R^{3)}{}_{ikl} + \delta^{(\alpha}_k R^{\beta)}{}_{ilj} + \delta^{(\alpha}_l R^{3)}{}_{ijk}. \tag{5}$$

In the matrix $\|\tilde{T}^{\alpha\beta}_{ij\,kl}\|$ we shall take combinations of upper indices as the column numbers, and the lower indices as the row numbers. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct numbers from 1 to n, then formula (5) shows that the determinant of the minor of order $(n-2)$ of the matrix corresponding to the column numbers $(\alpha_3\alpha_3)$, $(\alpha_3\alpha_4)$, ..., $(\alpha_3\alpha_n)$ and to the row numbers $(\alpha_1\alpha_3, \alpha_1\alpha_2)$, $(\alpha_1\alpha_4, \alpha_1\alpha_2)$, ..., $(\alpha_1\alpha_n, \alpha_1\alpha_2)$ is

$$\Delta = 2\left(R^{\alpha_3}{}_{\cdot\,\alpha_1\alpha_1\alpha_2}\right)^{n-2}. \tag{6}$$

For the determinant of the minor of order $(n-2)$ of the same matrix corresponding to the column numbers $(\alpha_1\alpha_4)$, $(\alpha_4\alpha_4)$, ..., $(\alpha_4\alpha_n)$ and to the row numbers $(\alpha_1\alpha_1, \alpha_2\alpha_3)$, $(\alpha_1\alpha_4, \alpha_2\alpha_3)$, ..., $(\alpha_1\alpha_n, \alpha_2\alpha_3)$, we obtain the formula

$$D_1 = 2\left(R^{\alpha_4}{}_{\cdot\,\alpha_1\alpha_2\alpha_3}\right)^{n-2}. \tag{7}$$

Hence it follows

Lemma 1. If the rank of $\left\|\tilde{T}^{\alpha\beta}_{ijkl}\right\|\rho < n-2$, then V_n is a space of constant curvature.

A joint consideration of relations B^1 and B^2 leads to the lemma:

Lemma 2. For V_n distinct from spaces of constant curvature, λ_i and μ are determined uniquely in terms of the objects V_n, the components of the tensor g_{ij}, and one of the components of the vector λ_i.

Lemmas 1 and 2 give the theorem:

Theorem 5. The totality of all Riemannian spaces $\bar{V}_n$ admitting a nontrivial geodesic mapping onto V_n, distinct from a space of constant curvature, depends on
$r \leq n(n+1)/2 - (n-4)$ essential parameters.

In a similar way, from B^1, B^2, and (3) ([2]) it follows

Theorem 6. The totality of all Riemannian spaces $\bar{V}_n$ admitting a nontrivial geodesic mapping onto V_n, distinct from an Einstein space, depends on
$r \leq n(n+1)/2 - (n-2)$ essential parameters.

3. Finally, the displacement vector ξ^i of a nontrivial infinitesimal geodesic transformation of V_n satisfies the equations ([3])

$$h_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{kj} + \psi_j g_{ki},$$

where $h_{ij} = \xi_{i,j} + \xi_{j,i}$, $\psi_i \neq 0$ is a gradient vector. Therefore the tensor $\hat{g}_{ij} = h_{ij} - 2\psi g_{ij}$ satisfies the equations A_1, and V_n admits a nontrivial geodesic mapping. On this basis, Theorems 5 and 6 give an estimate of the number of essential parameters on which the displacement vector of a general nontrivial infinitesimal geodesic transformation of V_n depends. Namely, in the first case $\tilde{r} \leq n^2 + 4$, and in the second $\tilde{r} \leq n^2 + 2$ (cf. ([4])).

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Note: Figure translations are in progress. See original paper for figures.