CARTAN PSEUDOGROUPS AND \(p\)-LIE ALGEBRAS

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Abstract

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CARTAN PSEUDOGROUPS AND p-LIE ALGEBRAS

1. In a paper of 1909 ([1]) Cartan classified all simple transitive pseudogroups of transformations. For a modern exposition of part of Cartan' s results see ([2–4]). In this exposition the theoretic-group question is replaced by the equivalent question of Lie algebras $\mathcal{L}$ satisfying the following conditions:

a) $\mathcal{L}$ is a topological Lie algebra with a linear topology defined by subspaces of finite codimension; $\mathcal{L}$ is complete in this topology.

b) In $\mathcal{L}$ there exists such an (open) subalgebra $\mathcal{L}_0$ that the system of subalgebras $\mathcal{L}_k$, defined by the conditions

$$\mathcal{L}_{-1} = \mathcal{L}; \qquad \mathcal{L}_{k+1} = \{x \in \mathcal{L}_k \mid [\mathcal{L}, x] \subset \mathcal{L}_k\}, \quad k \geq 0,$$

forms a complete system of neighborhoods of zero.

In what follows we shall call algebras satisfying conditions a), b) **infinite-dimensional Lie algebras**. It follows from the definition that $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$; $\mathcal{L}_k$ is an ideal in $\mathcal{L}_0$, $k > 0$. In view of this, a representation Γ of the finite-dimensional algebra $L_0 = \mathcal{L}_0/\mathcal{L}_1$ in the finite-dimensional module $V = \mathcal{L}_{-1}/\mathcal{L}_0$ is given.

In the papers mentioned it is proved that if $\mathcal{L}$ is a simple Lie algebra over an algebraically closed field k of characteristic zero and the representation Γ is irreducible, then $\mathcal{L}$ belongs to one of the series indicated below. The algebras of all series are realized in the ring of differential operators $\mathcal{D}$ of the ring $K = k[[x_1, \dots, x_n]]$ of formal power series over the field k,

$$\mathcal{D} = \sum_1^n f_i \frac{\partial}{\partial x_i}, \qquad f_i \in K. \tag{1}$$

The series are as follows:

1) $\mathcal{L}$ consists of all operators of the form (1) (the general algebra).

2) $\mathcal{L}$ consists of all operators leaving invariant the differential form $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ (the special algebra).

3) $n = 2m$; $\mathcal{L}$ consists of all operators leaving invariant the differential form

$$\omega = \sum_1^m dx_i \wedge dx_{m+i}$$

(the Hamiltonian algebra).

Cartan also proved that all simple infinite-dimensional Lie algebras are exhausted by series 1)–3) and by the series

4) $n = 2m + 1$; $\mathcal{L}_m$ consists of all operators multiplying the form

$$\omega = dx_n + \sum_{i=1}^m (x_i dx_{m+i} - x_{m+i} dx_i)$$

by an element of the ring K (the contact algebra).

The present note is devoted to the consideration of analogues of Cartan' s algebras over an algebraically closed field k of characteristic $p > 0$. Their definition remains fully meaningful; however, not one of them remains simple. Namely, in $\mathcal{L}$ there is an ideal J consisting of all operators of the form (1) for which $f_i \in K^p = K \cdot (x_1^p, \ldots, x_n^p)$. The quotient algebra $\mathcal{L}/J$ for all four series has finite dimension over k and is already very close to being simple. More precisely, in case 1) the algebra $\mathcal{L}/J = W_n$ is simple and $\dim W_n = np^n$, $(n, p) \neq (1, 2)$; in case 2) the commutant S_n of the algebra

$\mathcal{L}/J$ and $\dim S_n = (n-1)(p^n - 1)$, $n > 2$; in case 3) there is a simple ideal H_n of codimension 1 in the commutant of the algebra $\mathcal{L}/J$, $\dim H_n = p^n - 2$ ($p > 2$ or $n > 2$). Finally, in case 4) there is a simple algebra K_n, where $K_n = \mathcal{L}/J$ if $n + 3 \not\equiv 0(p)$, and $K_n = (\mathcal{L}/J, \mathcal{L}/J]$ if $n + 3 \equiv 0(p)$, $p > 2$. Accordingly, $\dim K_n = p^n$ or $p^n - 1$.

The algebras W_n, S_n, H_n, and K_n, for which we again retain the names general, special, Hamiltonian, and contact, are Lie p-algebras. In another form they had already been described earlier (see ([5–9])). The algebras of these series we shall also call **algebras of Cartan type**, and the simple p-algebras obtained by reduction mod p from simple Lie algebras over a field of characteristic zero, **classical**. As was said above, all algebras of Cartan type are realized as algebras of derivations of the truncated polynomial ring $k[x_1, \ldots, x_n]$, $x_i^p = 0$.

2. It seems probable to us that **the algebras of Cartan type together with the classical ones exhaust all simple Lie p-algebras** ($p > 5$). In any case, no other examples are apparently known.

Our main result proves only this conjecture under certain additional restrictions.

Theorem 1. *Let $\mathcal{L}$ be a simple Lie p-algebra over an algebraically closed field k of characteristic $p > 7$, possessing a proper subalgebra $\mathcal{L}_0$ such that* $\dim V < p$, $\dim L_0 < p$, *and the representation* Γ *is irreducible. Then* $\mathcal{L}$ *is either a classical algebra or an algebra of Cartan type of one of the series 1)—3). (The definition of* V, L_0, *and* Γ *was given in item 1.)*

We shall dwell on the main stages of the proof of Theorem 1. It proceeds in parallel with the proof of the corresponding theorem for infinite-dimensional Lie algebras over a field of characteristic zero (see ([2,3])), with the overcoming of certain difficulties caused by the finiteness of the characteristic. The elegant proof available in ([3]) is inapplicable in our case, since it uses the infinite-dimensionality of the algebra.

From the subalgebra $\mathcal{L}_0$ one constructs, as indicated in item 1, a filtration $\{\mathcal{L}_k\}$. From the irreducibility condition for the representation Γ it follows that all the terms $\mathcal{L}_k$ of the filtration will be p-subalgebras. Two cases are distinguished.

1) $\mathcal{L}_k = 0$ for $k \geq 2$;

2) $\mathcal{L}_2 \neq 0$.

In case 1) it is proved that $\mathcal{L}$ is an algebra of classical type. This follows from results obtained earlier by one of the authors ([10,11]). In case 2) it is proved that $\mathcal{L}$ is of Cartan type. First of all, it is established that L_0 is a direct sum of algebras of classical type and, possibly, a one-dimensional center. This is obvious over a field of characteristic zero, but nontrivial in our case. The result obtained makes it possible to apply to the study of the representation Γ Cartan' s theory ([12]) of p-representations of algebras of classical type.

Proposition. *The difference of any two weights of the representation* Γ *is expressible as the difference of certain roots of the algebra* L_0.

This assertion is a strengthening of Lemma 5.5 from ([4]). One can prove that the conditions of the proposition are satisfied only by the algebras A_n, $A_n + k$, C_n, $C_n + k$ and their standard representations. The considerations set forth already make it possible to determine uniquely, from L_0, the graded algebra $L = V + L_0 + L_1 + \cdots$ associated with the filtration $\{\mathcal{L}_k\}$. The coincidence of $\mathcal{L}$ and L is proved in the same way as in ([2]), since no restrictions on the characteristic arise here.

3. Simple Lie p-algebras of Cartan type possess a fundamental distinction from the classical ones. Namely, in all of them there exist proper subalgebras defined in an absolutely invariant manner, i.e., invariant under all automorphisms of the algebra. Further we shall

call such subalgebras invariant. In an algebra over a field of characteristic zero, an invariant subalgebra, of course, must be an ideal, and therefore a simple algebra cannot have a proper invariant subalgebra. Similarly, a simple algebra of classical type over a field of characteristic $p > 0$ cannot have them either.

One may state the following conjecture: if in a simple p-Lie algebra $\mathcal{L}$ $(p > 5)$ there is no proper invariant subalgebra, then $\mathcal{L}$ is of classical type. Of course, this assertion follows from the classification conjecture formulated at the beginning of § 2.

Let $\mathcal{L}$ be a simple Cartan p-Lie algebra. In it there exists an important invariant subalgebra, which we shall denote by $\mathfrak{C}$. The subalgebra $\mathfrak{C}$ is generated by all elements $c \in \mathcal{L}$ for which $(\operatorname{ad} c)^2 = 0$. The invariance of $\mathfrak{C}$ is obvious from the definition. For the general properties of the algebra $\mathfrak{C}$ see ([10]). Direct verification shows that in algebras of Cartan type $\mathfrak{C}$ is a proper subalgebra. Moreover, for an arbitrary algebra $\mathcal{L}$ the subalgebra $\mathfrak{C}$ is nonzero if and only if in $\mathcal{L}$ there is such a subalgebra $\mathcal{L}_0$ that, in the filtration $\{\mathcal{L}_k\}$ determined by it (see § 1), the term $\mathcal{L}_2 \neq 0$.

Theorem 2. *In a Lie algebra $\mathcal{L}$ of Cartan type there exists a unique proper maximal invariant subalgebra $\mathcal{L}_0$, coinciding with the normalizer $N_L(\mathfrak{C})$ of the subalgebra $\mathfrak{C}$. When $\mathcal{L}$ is realized as an algebra of differential operators of a truncated polynomial ring, $\mathcal{L}_0$ consists of all operators of the form (1) for which $f_i \in (x_1, \dots, x_n)$. In the filtration $\{\mathcal{L}_k\}$ determined by the subalgebra $\mathcal{L}_0$, the term $\mathcal{L}_k$ consists of operators of the form (1) for which $f_i \in (x_1, \dots, x_n)^{k+1}$.*

For the proof of the classification conjecture it would be extremely important to have the following assertion: in every simple p-Lie algebra $\mathcal{L}$, different from a classical one, there exists a unique proper maximal invariant subalgebra. It coincides with the normalizer $N_L(\mathfrak{C})$ of the subalgebra $\mathfrak{C}$.

The validity of this conjecture would make it possible to choose the most natural filtration in an arbitrary simple p-Lie algebra. The point is that the investigation is complicated by the presence of a number of pathological filtrations different from the filtration described in Theorem 2. For example, in the Hamiltonian algebra H_2 there exists a filtration $\{\mathcal{L}'_k\}$ with last term $\mathcal{L}'_{p-2} \neq 0$, in which the algebra $\mathcal{L}'_0$ is the general algebra W_1, and the representation Γ is an irreducible representation of this algebra of degree $p-1$.

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Note: Figure translations are in progress. See original paper for figures.