Reports of the Academy of Sciences of the USSR

1966

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1966. Volume 166, No. 3

MATHEMATICS

V. M. Kalinin

ASYMPTOTIC EXPANSIONS FOR FREQUENTLY OCCURRING PROBABILITY DISTRIBUTIONS

(Presented by Academician Yu. V. Linnik on 10 V 1965)

In the present article, as in papers ($^{1-6}$), the aim is to refine limit theorems on the convergence of the binomial and multinomial distributions to the normal and Poisson distributions, of Student' s distribution to the normal distribution, etc. However, the main goal is not the estimation of the remainder term, but the derivation of general formulas for the coefficients in asymptotic expansions. The principal results are formulated in the following six theorems.

Theorem 1. Let $\lambda_1, \ldots, \lambda_{k-1}$ be fixed positive numbers; $m_1, \ldots, m_{k-1}$ fixed nonnegative integers; $\lambda = \lambda_1 + \cdots + \lambda_{k-1}$, $m = m_1 + \cdots + m_{k-1}$, $m_k = n - m$. Then for any $k = 2, 3, \ldots$ the following asymptotic* expansion holds as $n \to \infty$:

$$\frac{n!}{m_1! \cdots m_k!} (\lambda_1/n)^{m_1} \cdots (\lambda_{k-1}/n)^{m_{k-1}} (1 - \lambda/n)^{m_k} =$$

$$= \left(\prod_{i=1}^{k-1} \frac{\lambda_i^{m_i} e^{-\lambda_i}}{m_i!} \right) \exp \left[\sum_{j=1}^{\infty} \frac{\mathcal{L}_j(m, \lambda)}{n^j} \right] = \left(\prod_{i=1}^{k-1} \frac{\lambda_i^{m_i} e^{-\lambda_i}}{m_i!} \right) \left[1 + \sum_{j=1}^{\infty} \frac{M_j(m, \lambda)}{h^j} \right],$$

where

$$\mathcal{L}_j(m, \lambda) = \frac{B_{j+1} - B_{j+1}(m)}{j(j+1)} + \frac{\lambda^j m}{j} - \frac{\lambda^{j+1}}{j+1},$$

$$M_j(m, \lambda) = \sum \frac{\mathcal{L}_1^{\nu_1}(m, \lambda) \cdots \mathcal{L}_j^{\nu_j}(m, \lambda)}{\nu_1! \cdots \nu_j!}$$

is a polynomial of degree $2j$ in m and λ [here and below, where the summation limits are not indicated, it is assumed that the sum is taken over all nonnegative integers $\nu_1, \dots, \nu_j$ satisfying the equation $\nu_1 + 2\nu_2 + \dots + j\nu_j = j$]; $B_{j+1}(x)$ is the Bernoulli polynomial of degree $j+1$; $B_{j+1} = B_{j+1}(0)$ are the Bernoulli numbers.

The theorem remains valid if $\lambda_i = \lambda_i(n)$ are nonnegative sequences bounded as $n \to \infty$.

Theorem 2. Let λ be a fixed positive number, m a fixed nonnegative integer. As $n \to \infty$, the following asymptotic expansion holds:

$$\sum_{i=0}^{m} C_n^i \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = F(m) + \sum_{j=1}^{\infty} \frac{V_j(m, \lambda)}{n^j},$$

where

$$F(x) = \begin{cases} \sum\limits_{i=0}^{m} \dfrac{\lambda^i e^{-\lambda}}{i!}, & x = m, \\ 0, & x < 0; \end{cases}$$

* In the sense that the remainder term of the series in the exponent has the order of the first omitted term.

$$V_j(m, \lambda) = \sum_{\nu=0}^{2j} \frac{\lambda^\nu}{\nu!} F(m-\nu) \sum_{\mu=0}^{\nu} (-1)^\mu C_\nu^\mu M_j(\nu - \mu, \lambda).$$

Theorem 3. As $\lambda \to \infty$, the following asymptotic expansion is valid uniformly* with respect to y in any finite interval $a \le y \le b$:

$$\frac{\lambda^m e^{-\lambda}}{m!} = \frac{e^{-y^2/2}}{\sqrt{2\pi\lambda}} \exp\left[\sum_{j=1}^{\infty} \frac{W_j(y)}{(\sqrt{\lambda})^j}\right],$$

where

$$y = (m - \lambda + \theta)/\sqrt{\lambda},$$

θ is an arbitrary real (as are all other quantities in this paper) fixed number,

$$W_j(y) = (-1)^{j+1} \sum_{\nu=0}^{1+[j/2]} \frac{C_{j-\nu+2}^\nu y^{j-2\nu+2} B_\nu(\theta)}{(j-\nu+1)(j-\nu+2)}.$$

Theorem 4. Let $p_1, \dots, p_k$ be fixed positive numbers, with $p_1 + \dots + p_k = 1$; let $m_1, \dots, m_k$ be positive integers, $m_1 + \dots + m_k = n$. Define y_i by the formula

$$y_i = (m_i - np_i)/\sqrt{np_{iq}i},$$

$$q_i = 1 - p_i, \quad i = 1, \dots, k.$$

Then, uniformly for all m_i for which the y_i lie in arbitrary finite intervals $a_i \le y_i \le b_i$, the following asymptotic expansion is valid as $n \to \infty$:

$$\frac{n!}{m_1! \cdots m_k!} p_1^{m_1} \cdots p_k^{m_k} = \frac{\exp\left[-\frac{1}{2}\sum_{i=1}^{k} q_i y_i^2\right]}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1 \cdots p_k}} \exp\left[\sum_{j=1}^{\infty}\sum_{i=1}^{k}\frac{G_{ji}}{(\sqrt{np_i})^j}\right],$$

where

$$G_{ji} = \begin{cases} (-1)^{j+1}\sum_{\nu=0}^{[(j+1)/2]}\dfrac{B_\nu C_{j-\nu+2}^{\nu}(y_i\sqrt{q_i})^{j-2\nu+2}}{(j-\nu+1)(j-\nu+2)}, & j \ne 4\mu - 2, \\ -\sum_{\nu=0}^{2\mu-1}\dfrac{B_\nu C_{4\mu-\nu}^{\nu}(y_i\sqrt{q_i})^{4\mu-2\nu}}{(4\mu-\nu-1)(4\mu-\nu)} - \dfrac{(1-p_i^{2\mu})}{2\mu(2\mu-1)}B_{2\mu}, & j = 4\mu - 2. \end{cases}$$

For $k = 2$ we obtain

$$C_n^m p^m q^{n-m} = \frac{e^{-y^2/2}}{\sqrt{2\pi npq}} \exp\left[\sum_{j=1}^{\infty}\frac{G_j(y)}{(\sqrt{npq})^j}\right]$$

$$= \frac{e^{-y^2/2}}{\sqrt{2\pi npq}}\left[1 + \sum_{j=1}^{\infty}\frac{T_j(y)}{(\sqrt{npq})^j}\right],$$

where

$$y = (m - np)/\sqrt{npq},$$

$$G_j(y) = \begin{cases} -\sum_{\nu=0}^{[(j+1)/2]}\dfrac{B_\nu C_{j-\nu+2}^{\nu} y^{j-2\nu+2}}{(j-\nu+1)(j-\nu+2)}\left[p^{j-\nu+1} + (-1)^j q^{j-\nu+1}\right], & j \ne 4\mu - 2, \\ -\sum_{\nu=0}^{2\mu-1}\dfrac{B_\nu C_{4\mu-\nu}^{\nu} y^{4\mu-2\nu}\left(p^{4\mu-\nu-1} + q^{4\mu-\nu-1}\right)}{(4\mu-\nu-1)(4\mu-\nu)} & \\ -\dfrac{B_{2\mu}}{2\mu(2\mu-1)}\left[p^{2\mu-1} + q^{2\mu-1} - (pq)^{2\mu-1}\right], & j = 4\mu - 2, \end{cases}$$

$$T_j(y) = \sum \frac{G_1^{\nu_1}(y) \cdots G_j^{\nu_j}(y)}{\nu_1! \cdots \nu_j!}$$

is a polynomial of degree $3j$, whose parity coincides with the parity of j.

Theorem 5. As $n \to \infty$, the asymptotic expansion holds

* Uniformity refers to the choice of the constant in the O-estimates.

for arbitrary fixed θ, uniformly with respect to $a \leqslant y_1 \leqslant y_2 \leqslant b$,

$$\sum_{i=\mu+1}^{\mu+m} C_n^i p^i q^{n-i} = \int_{y_1}^{y_2} \varphi(y)\,dy + \sum_{j=1}^{\infty} \frac{Q_j}{(\sqrt{npq})^j},$$

where

$$Q_j = -\frac{B_j(\theta)}{j!} H_{j-1}(y)\varphi(y)\Big|_{y_1}^{y_2} + \int_{y_1}^{y_2} \varphi(y) T_j(y)\,dy +$$

$$+ \sum_{\nu=1}^{j-1} \frac{(-1)^\nu B_\nu(\theta)}{\nu!} \frac{d^{\nu-1}}{dy^{\nu-1}} \varphi(y) T_{j-\nu}(y)\Big|_{y_1}^{y_2},$$

$$y_1 = (\mu + \theta - np)/\sqrt{npq}, \qquad y_2 = (\mu + m + \theta - np)/\sqrt{npq},$$

$$\varphi(y) = e^{-y^2/2}/\sqrt{2\pi}.$$

$H_{j-1}(y)$ is the Hermite polynomial of degree $j-1$:

$$H_{j-1}(y) = (-1)^{j-1} e^{y^2/2} \times \frac{d^{j-1}}{dy^{j-1}} e^{-y^2/2}.$$

Theorem 6. Uniformly with respect to y in any finite interval $a \leqslant y \leqslant b$, the following asymptotic expansion holds as $n \to \infty$:

$$\frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} (1 + y^2/n)^{-\frac{n+1}{2}} = \varphi(y) \exp\left[\sum_{j=1}^{\infty} \frac{K_j(y)}{n^j}\right] =$$

$$= \varphi(y) \left[1 + \sum_{j=1}^{\infty} \frac{P_j(y)}{n^j}\right],$$

where

$$K_j(y) = \frac{(-1)^j}{2} \left(\frac{y^{2j}}{j} - \frac{y^{2j+2}}{j+1}\right) - \frac{2^{j+1}-1}{j(j+1)} B_{j+1},$$

$$P_j(y) = \sum \frac{K_1^{\nu_1}(y) \cdots K_j^{\nu_j}(y)}{\nu_1! \cdots \nu_j!}$$

is an even polynomial of degree $4j$.

Received
7 V 1965

References Cited

1. S. N. Bernstein, *Izv. AN SSSR, ser. matem.*, 7, No. 1, 3 (1943).

2. A. N. Kolmogorov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, 12 (1945).

3. Yu. V. Prokhorov, *UMN*, 8, issue 3, 136 (1953).

4. J. V. Uspensky, *Introduction to Mathematical Probability*, N. Y., 1937.

5. W. Feller, *Ann. Math. Statistics*, 16, No. 4, 319 (1945).

6. W. Wasow, *Proc. Symposia in Appl. Math.*, 6, 1956.

Note: Figure translations are in progress. See original paper for figures.