# ON THE SOLUTION OF FUNCTIONAL EQUATIONS BY THE METHOD OF REGULARIZATION

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE SOLUTION OF FUNCTIONAL EQUATIONS BY THE METHOD OF REGULARIZATION

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**1.** Let $U$ and $F$ be two functional spaces. Consider the problem $R$ of determining a solution $u \in U$ of the equation

$$Au = f \tag{1}$$

for a given function $f \in F$, where $A[U] \to F$ and the equality $Au_1 = Au_2$ holds only in the case $u_1 = u_2$.

**Definition.** If the problem $R$ has a solution and, for every given function $f_\delta \in F$, $0 < \|f - f_\delta\|_F \leq \delta$, one can indicate at least one function $\tilde{f}_\delta \in F$ such that: a) the equation $Aw = \tilde{f}_\delta$ has a solution $\widetilde{w}_\delta = R[\tilde{f}_\delta]$; b) for every $\varepsilon > 0$ one can indicate $\delta_0 = \delta_0(\varepsilon) > 0$ such that for all $\delta$, $0 < \delta \leq \delta_0(\varepsilon)$, the inequality $\|\widetilde{w}_\delta - u\|_U < \varepsilon$ holds, then we shall say that the problem $R$ admits a stable solution.

We note that a well-posed problem always admits a stable solution. An algorithm (method) realizing a stable solution of problem (1) will be called stable.

The following theorem establishes the connection between the concept of a stable method for solving problem (1) and the concept of a regularizing algorithm ([1]).

**Theorem 1.** *In order that problem (1) admit a stable solution, it is necessary and sufficient that it be regularizable.*

We next make the spaces $U$ and $F$ more concrete. Let $H$ be a Hilbert space and let $L$ be a certain positive definite self-adjoint operator with domain of definition $D_L \subset H$. Denote by $H_L$ ([2]) the completion of $D_L$ in the sense of the scalar product

$$(v, w)_L = (Lv, w), \qquad v, w \in D_L.$$

Then $F = H$, $U = H_L$. The mapping $A[H_L] \to H$ will be assumed linear.

**2.** Consider the following variational problem: find a function $\widetilde{w}^\alpha \in H_L$ delivering the minimum value of the functional

$$\Phi^\alpha[w;\tilde{f}] = \|Aw - \tilde{f}\|^2 + \alpha\|w\|_L^2, \qquad w \in H_L, \tag{2}$$

where $\alpha > 0$ is a parameter.

We note that functionals of the indicated form were considered earlier in the works ([1,3–7]).

**Theorem 2.** *For any function $\tilde{f} \in H$, the solution of problem (2) exists and is unique for every $\alpha > 0$.*

The proof of this theorem is based on the identity

$$\Phi^\alpha\left[\frac{u' - u''}{2}; 0\right] = \frac{1}{2}\Phi^\alpha[u';\tilde{f}] + \frac{1}{2}\Phi^\alpha[u'';\tilde{f}] - \Phi^\alpha\left[\frac{u' + u''}{2};\tilde{f}\right],$$

$$u', u'' \in H_L,$$

and is carried out as in ([5]).

**Theorem 3.** *Let $w^\alpha$ be the solution of problem (2) for $\tilde{f} = f$, and $w^{\alpha,\delta}$ the solution of the same problem for $\tilde{f} = f_\delta$.*

Then the a priori estimate is valid

$$\|w^{\alpha,\delta} - u\|_L \le \omega(\alpha; u) + \delta/\sqrt{\alpha}; \tag{3}$$

where $\omega(\alpha; u) = \|u - w^\alpha\|_L \to 0$, as $\alpha \to 0$.

3. Theorems 2 and 3 show the fundamental possibility of obtaining a sufficiently accurate approximation to the solution of equation (1) in the form of a solution of problem (2) for a suitably chosen value of the parameter $\alpha$.

We shall next consider and prove the possibility of choosing such values of the parameter by the magnitude of the residual

$$\rho_\delta(\alpha) = \|Aw^{\alpha,\delta} - f_\delta\|^2.$$

**Lemma.** Let $0 < \delta < \|f\|/2$ and let the constant $c$ be such that

$$1 < c < (\|f\| - \delta)^2/\delta^2.$$

Then there exists a unique value of the parameter $\alpha = \alpha_c$ for which $\rho_\delta(\alpha_c) = c\delta^2$.

**Theorem 4.** If $\delta \to 0$, then $w^\delta = w^{\alpha_c,\delta} \to u$ in $H_L$.

Indeed, for any $\alpha > 0$ we have

$$\Phi^\alpha[w^{\alpha,\delta}; f_\delta] \le \Phi^\alpha[u; f_\delta] \le \delta^2 + \alpha\|u\|_L^2.$$

Putting here $\alpha = \alpha_c$, we have

$$c\delta^2 + \alpha_c\|w^\delta\|_L^2 \le \delta^2 + \alpha_c\|u\|_L^2,$$

whence we obtain

$$\|w^\delta\|_L^2 \le \|u\|_L^2. \tag{4}$$

On the other hand,

$$\lim_{\delta\to 0}\|Aw^\delta - f\|^2 = \lim_{\delta\to 0}\|Aw^\delta - f_\delta\|^2 = \lim_{\delta\to 0} c\delta^2 = 0$$

and, consequently, $w^\delta$ has a unique weak limit point, which is the function $u$.

From this fact and estimate (4) it follows that $\lim_{\delta\to 0}\|w^\delta\|_L = \|u\|_L$. It remains to note that in Hilbert space weak convergence together with convergence of the norms implies strong convergence. Theorem 4 is proved.

4. The results obtained are applicable to a sufficiently broad class of ill-posed problems. As an example we indicate the class of problems described by integral equations of the form

$$A[u] \equiv A(x)u(x) + \int_a^b K(x,\xi)u(\xi)\,d\xi = f(x), \qquad a \le x \le b, \tag{5}$$

where $f \in L_2[a,b]$; $K(x,\xi)$ is a function continuous in the square $a \le x \le b,\ a \le \xi \le b$; $A(x)$ is a continuous function that vanishes on some nonempty set $E = \{x;\ A(x) = 0\}$. For $A(x) \equiv 0$, equation (5) was considered in papers ([1],[3]).

Let $u \in W_2^{(n)}[a,b]$ ([8]) be a solution of equation (5), where the norm in $W_2^{(n)}[a,b]$ is defined by the relation

$$\|u\|^2_{W_2^{(n)}[a,b]} = \int_a^b ((u^{(n)})^2 + qu^2)\,dx,$$

where $q(x) > 0$ is a continuous function.

Define the operator $L$ on the class of $2n$-times differentiable functions by the relation

$$Lw \equiv (-1)^n w^{(2n)} + qw, \qquad w^{n+i}(a) = w^{n+i}(b) = 0, \qquad i = 0, 1, \dots, n-1,$$

$$a \leq x \leq b.$$

Then $\|w\|_L = \|w\|_{W_2^{(n)}[a,b]}$, and, according to what has been proved, one can obtain a solution of problem (2) close to the solution of equation (5) in $W_2^{(n)}[a,b]$.

5. Let $u \in H_L$, and let the functions $f_\delta \in H$ be such that $\|u - f_\delta\| < \delta$. It is required to give an algorithm $\overline{R}$ such that: a) $u_\delta = \overline{R}[f_\delta] \in H_L$; b) for any prescribed $\varepsilon > 0$ one can find $\delta_0 = \delta_0(\varepsilon)$ such that $\|u_\delta - u\|_L < \varepsilon$ for all $\delta$, $0 < \delta \leq \delta_0$. We shall call this problem the **reconstruction problem**.

Define the functional

$$\overline{\Phi}^\alpha[w; f_\delta] = \|w - f_\delta\|^2 + \alpha\|w\|_L^2, \qquad w \in H_L, \tag{6}$$

and let $\overline{w}_{\alpha_c,\delta}$ be the solution of problem (6), defined by the discrepancy principle.

**Theorem 5.** *The algorithm $\overline{R}$ for determining $\overline{w}_{\alpha_c,\delta}$ is a stable method for solving the reconstruction problem.*

Theorem 5 makes it possible to construct a generalized scheme of the least-squares method [9], frequently used in the processing of experimental information, which is more stable computationally than the usual one and gives an approximation, on the average, not only to the function itself but also to its derivatives.

If $f_\delta \in H$ is given in the form of a Fourier series in the eigenfunctions of the operator $L$, then Theorem 5 yields a stable method of summing this series in $H_L$.

Among results close to the one formulated, we note [10].

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*Note: Figure translations are in progress. See original paper for figures.*