SOLUTION OF THE HARDY-LITTLEWOOD PROBLEM AND OF ITS INDEFINITE ANALOGUE IN SECTORS AND CONTOURS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 511

MATHEMATICS

A. A. POLYANSKII

SOLUTION OF THE HARDY-LITTLEWOOD PROBLEM AND OF ITS INDEFINITE ANALOGUE IN SECTORS AND CONTOURS

(Presented by Academician Yu. V. Linnik on 5 VIII 1965)

1. The dispersion method, created by Yu. V. Linnik (^{1}) and perfected by B. M. Bredikhin (^{3}), has proved very effective in the solution of some binary additive problems.

In particular, by means of the dispersion method Yu. V. Linnik solved (^{2}) the Hardy-Littlewood problem, which consists in finding an asymptotic formula for the number of solutions $Q(n)$ of the equation

$$p + \xi^2 + \eta^2 = n. \tag{1}$$

B. M. Bredikhin solved (^{4}) the indefinite analogue of the Hardy-Littlewood problem, which consists in finding an asymptotic formula for the number of solutions $S(n)$ of the equation

$$p - \xi^2 - \eta^2 = l. \tag{2}$$

In equations (1) and (2), ξ, η independently run through the integers under the condition $0 < \xi^2 + \eta^2 \leq n$ (in equation (1) this condition is fulfilled automatically), p runs through the primes, l is a fixed nonzero integer; n is a sufficiently large natural number (the main parameter of the paper).

In this note we consider theorems giving lower estimates for the number of solutions of equations (1) and (2) in sectors (and contours). The proof is based on combining the dispersion method with unimodular transformations of the quadratic form

$$\varphi(\xi, \eta) = \xi^2 + \eta^2.$$

2. Let $Q_{\Delta\varphi}(n)$ be the number of solutions of equation (1) under the condition that $(\xi, \eta) \in (\Delta\varphi, \sqrt{n})$, where $(\Delta\varphi, \sqrt{n})$ is a circular sector of radius $\sqrt{n}$ with given aperture angle $\Delta\varphi = \varphi_2 - \varphi_1$, $0 \leq \varphi_1 < \varphi_2 \leq 2\pi$; let $S_{\Delta\varphi}(n)$ be the number of solutions of equation (2) under the same conditions. Let, further, ε be a given small number, $0 < \varepsilon < (\varphi_2 - \varphi_1)/2\pi$; b be some integer, relatively prime to l and n, depending on $\varepsilon, \Delta\varphi, n$, and in the last case $b = O((\ln n)^K)$, where $K = K(\varepsilon, \Delta\varphi)$ is a positive constant.

Theorem 1. *As* $n \to \infty$,

$$Q_{\Delta\varphi}(n) \geq \left(\frac{\varphi_2 - \varphi_1}{2\pi} - \varepsilon\right)\frac{1}{b}\,Q(n). \tag{3}$$

Theorem 2. *As* $n \to \infty$,

$$S_{\Delta\varphi}(n) \geq \left(\frac{\varphi_2 - \varphi_1}{2\pi} - \varepsilon\right)\frac{1}{b}\,S(n). \tag{4}$$

As is known (see ($^{2-4}$)),

$$Q(n) = \pi A_0 \frac{n}{\ln n} \prod_{p/n} \frac{(p-1)(p-\chi_4(p))}{p^2 - p + \chi_4(p)} + R(n), \tag{5}$$

$$S(n) = \pi A_0 \frac{n}{\ln n} \prod_{p/l} \frac{(p-1)(p-\chi_4(p))}{p^2 - p + \chi_4(p)} + R(n), \tag{6}$$

where

$$A_0 = \prod_{p>2}\left(1 + \frac{\chi_4(p)}{p(p-1)}\right),$$

$\chi_4(m)$ is the nonprincipal character modulo 4, and $R(n) = O(n(\ln n)^{-1{,}042})$.

Let us note the obvious corollaries following from Theorems 1 and 2, which provide new facts in the theory of prime numbers.

Corollary 1. Every sufficiently large number n is representable in the form

$$n = p + \xi^2 + \eta^2,$$

where (ξ, η) belongs to the angle of aperture $\Delta\varphi$.

Corollary 2. There exist infinitely many prime numbers of the form

$$p = \xi^2 + \eta^2 + l,$$

where (ξ, η) belongs to the angle of aperture $\Delta\varphi$.

We shall precede the proof of the theorems by a number of lemmas.

3. Consider a generalization of equations (1) and (2):

$$p + b(\xi^2 + \eta^2) = n, \tag{7}$$

$$p - b(\xi^2 + \eta^2) = l, \tag{8}$$

where $(b, n) = 1$, $(b, l) = 1$, $b(\xi^2 + \eta^2) \leq n$, and

$$b = O((\ln n)^K), \tag{9}$$

$K > 0$ being a constant.

Let $Q_b(n)$ denote the number of solutions of equation (7), and $S_b(n)$ the number of solutions of equation (8).

Lemma 1. As $n \to \infty$,

$$Q_b(n) = \frac{1}{b}\left[Q(n) \prod_{\substack{p/b \\ p\times 2n}} \frac{p^2}{p^2 - p + \chi_4(p)} + R(n)\right]. \tag{10}$$

Lemma 2. As $n \to \infty$,

$$S_b(n) = \frac{1}{b}\left[S(n) \prod_{\substack{p/b \\ p\times 2l}} \frac{p^2}{p^2 - p + \chi_4(p)} + R(n)\right]. \tag{11}$$

The proof of Lemma 1 is carried out by a variant of the dispersion method, based on the use of constructions of the expected number of solutions of certain auxiliary equations, and is obtained as a certain modification of the arguments from paper (3).

The proof of Lemma 2 is carried out by a variant of the dispersion method, based on the use of cotangent numbers, and is obtained as a certain modification of the arguments from paper (5).

The passage from the given binary equations (1) and (2) (in sectors) to equations (7) and (8) is based on the following lemma.

Lemma 3. Whatever the prescribed ε_0 (a small positive number) and an integer $m = O(n)$ may be, there exists a sector $(\delta\varphi_0, \sqrt{n})$ of the angle $\delta\varphi_0 = \varphi - \varphi_0$, where $0 < \varphi \leq \varepsilon_0$, $\varphi_0 = 0$. Moreover, $\cos\varphi = a/c$, $\sin\varphi = b/c$, where a, b, c^2 are positive integers, $(c^2, m) = 1$, and $c = O(\ln n)$.

For the proof it is sufficient to consider the sector $(\delta\varepsilon_0, \ln n)$, where $\delta\varepsilon_0 = \varepsilon_0 - 0$, in which, as follows from the results of I. P. Kubilyus (6), there is at least one Gaussian prime number $\mathfrak{p}$ with norm $N(\mathfrak{p}) = a^2 + b^2 = c^2$, where a, b, c will satisfy the requirements of the lemma for $\varphi = \arg\mathfrak{p}$.

4. Proof of Theorem 1. Choose ε_0 sufficiently small in comparison with $\Delta\varphi$. Using Lemma 3, cover the circle of radius $\sqrt{n}$, without overlap, by sectors of the form $(\delta\varphi_i; \sqrt{n})$, where $\delta\varphi_i = \varphi'_{i+1} - \varphi'_i = \varphi$, $\varphi'_0 = 0$, $\varphi'_1 = \varphi$, $i = 0, 1, 2, \dots, N-1$, $N = [2\pi/\varphi] + 1$. The last sector may be incomplete. The given sector $(\Delta\varphi, \sqrt{n})$ is thereby divided po-

covered by $m = [\Delta\varphi/\varphi] + \theta,\ 0 \le \theta \le 1$, partial sectors, where the two extreme sectors may be incomplete.

It is not difficult to see that $\cos\varphi'_i = a_i/c^{i+1}$, $\sin\varphi'_i = b_i/c^{i+1}$. In equation (7) put $b = c^{4N}$; then condition (9) will be satisfied.

Consider the number of solutions of equation (7) in the sector $(\delta\varphi_i, \sqrt{n})$, $i = 0, 1, \dots, N-1$, i.e., under the condition $(\xi, \eta) \in (\delta\varphi_i, \sqrt{n})$. In at least one of these sectors, whose number we denote by s, there will be no fewer than $\frac{1}{N}Q_b(n)$ solutions.

The point of the sector $(\delta\varphi_s, \sqrt{n})$, by means of a unimodular transformation of the form

$$S_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix} \tag{12}$$

will be successively transformed into m partial sectors of the given sector $(\Delta\varphi, \sqrt{n})$. In (12) the matrix elements are determined from the conditions

$$\cos(\varphi_s - \varphi_i) = a_j/c^j, \quad \sin(\varphi_s - \varphi_i) = b_j/c^j,$$

where i runs through the numbers of the indicated m partial sectors. We then obtain:

$$\frac{1}{N}Q_b(n) \le \sum_{\substack{p+b(\xi^2+\eta^2)=n \\ (\xi,\eta)\in(\delta\varphi_s,\sqrt{n})}} 1 = \sum_{\substack{p+b(x^2+y^2)=n \\ (x,y)\in(\delta\varphi_i,\sqrt{n})}} 1. \tag{13}$$

Note that in (13) integral (ξ, η) do not necessarily correspond to integral (x, y). From (13) it follows that

$$\frac{1}{N}Q_b(n) \le \sum_{\substack{p+x^2+y^2=n \\ (x,y)\in(\delta\varphi_i,\sqrt{n})}} 1, \tag{14}$$

where (x, y) now run over the integral points of the partial sector.

Summing (14) over all partial sectors and neglecting the extreme ones, we obtain

$$\frac{m-2}{N} Q_b(n) \leq Q_{\Delta\varphi}(n). \tag{15}$$

From (10) and (15), (3) follows. Thus the theorem is proved.

Analogously, with the aid of Lemmas 2 and 3, Theorem 2 is proved.

We note that the quantity ε in Theorems 1 and 2 is not necessary and, upon refining the arguments, may be eliminated, but this will entail an increase in b.

5. Using considerations of I. P. Kubilius (^{6}), Theorems 1 and 2 can be applied to obtain lower estimates for the number of solutions of equations (1) and (2) in homothetically expanding contours. Of interest here are piecewise-smooth convex contours passing through the origin and having the form

$$C: \ r = f(\varphi), \quad \varphi_1 \leq \varphi \leq \varphi_2; \quad 0 < \varphi_2 - \varphi_1 < \pi/2; \quad f(\varphi) \geq 0.$$

In conclusion I express my deep gratitude to Yu. V. Linnik for valuable advice and attention to my work.

Kuibyshev State
Pedagogical Institute
named after V. V. Kuibyshev

Received
2 VIII 1965

REFERENCES

[1] Yu. V. Linnik, *The dispersion method in binary additive problems*, L., 1961.
[2] Yu. V. Linnik, *Izv. AN SSSR, Ser. Mat.*, **24**, No. 5 (1960).
[3] B. M. Bredikhin, *UMN*, **20**, issue 3 (122) (1965).
[4] B. M. Bredikhin, *DAN*, **142**, No. 4 (1962).
[5] B. M. Bredikhin, *Izv. AN SSSR, Ser. Mat.*, **27**, No. 3 (1963).
[6] I. P. Kubilius, *Uch. Zap. LGU, Ser. Mat.*, **19** (1950).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.