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THE STRUCTURE OF FUNDAMENTAL MATRICES OF R-SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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For systems with periodic coefficients, the structure of fundamental matrices is described by Floquet' s theorem (see, for example, (1), p. 90, Theorem 5.1). A generalization of Floquet' s theorem to systems with quasiperiodic coefficients was made by A. E. Gelman (see (7,8)) (the case $n = 2$) and by L. Ya. Adrianova (see (2)) (the case of arbitrary n). Systems with almost periodic coefficients under certain special assumptions were studied by Lillo (see (9)) and by B. F. Bylov (see (3–6)).

In the present work a theorem is proved which generalizes Floquet' s theorem to R-systems with almost periodic coefficients (uniformly almost periodic, see (10)). For the proof, the method of (13) is used; moreover, the first part of the present work is devoted to the development of this method, and concerns arbitrary R-systems. The above-mentioned results and methods of reasoning of B. F. Bylov are also used.

Definition of an R-system. 1. Let the function $q(t)$ satisfy the Lipschitz condition. For each $\varepsilon > 0$ construct inductively T_i $(i = 1, 2, ...)$ as follows: $T_0 = 1$, $T_i =$ sup of the lengths of intervals not containing intervals $[\tau_1, \tau_2]$ such that $\tau_2 - \tau_1 \geq T_{i-1}$ and

$$\frac{q(\tau_2) - q(\tau_1)}{\tau_2 - \tau_1} \leq \lim_{t-\tau\to+\infty} \frac{q(t) - q(\tau)}{t - \tau} - \varepsilon.$$

We shall say that: a) $q(t) \in R_1$, if either some $T_i = +\infty$, or

$$\sum_{i=1}^{\infty} \frac{T_{i-1}}{T_i} = +\infty \quad (\varepsilon > 0 \text{ arbitrary});$$

b) $q(t) \in \widetilde{R}$, if $q(t) \in R_1$, $-q(t) \in R_1$.

2. $A(t) \in R$, if the following holds. Let $U^{(k)}(t)$ $(k = 1, 2, \ldots)$ be arbitrary Perron transformations which reduce the system $\dot{x} = A(t)x$ to triangular form $\dot{u} = B^{(k)}(t)u$, and let $t_k \to +\infty$ be an arbitrary sequence such that there exists, uniformly on intervals, the limit

$$q_i(t) = \lim_{k\to\infty} \int_{t_k}^{t_k+t} b_{ii}^{(k)}(\tau)\, d\tau \quad (b_{ii}^{(k)}(t) \text{ is the } i\text{-th diagonal element of } B^{(k)}(t)).$$

Then $q_i(t) \in \widetilde{R}$.

Lemma 1. *Let there be given a sequence of functions* $q_k(t) \equiv q(t) \in \widetilde{R}$ $(t \geq t_0)$ *such that*

$$|q_k(t) - q_k(\tau)| \leq a(t - \tau) \quad (k = 1, 2, \ldots;\ t \geq \tau \geq t_0).$$

Let

$$\lambda = \lim_{k\to\infty} \frac{q_k(t_k) - q_k(\tau_k)}{t_k - \tau_k}$$

for some sequence of intervals $[\tau_k, t_k]$, $t_k - \tau_k \xrightarrow[k\to\infty]{} +\infty$.

Then for every $\varepsilon > 0$ there exist a sequence $\theta_i \geq t_0$ and a sequence of indices k_i $(i = 1, 2, \ldots)$ such that

$$r_i(t) = q_{k_i}(\theta_i + t) - q_{k_i}(\theta_i) \longrightarrow q(t) \quad {}_{i\to\infty}$$

uniformly on each interval for $t \geq 0$, and for the function $q(t)$ one has

$$q(t) - q(\tau) \geq (\lambda - \varepsilon)(t - \tau) - d_\varepsilon$$

for any $t \geq \tau \geq 0$ and some $d_\varepsilon \geq 0$ (d_ε is a function of ε).

We give the main idea of the proof. Suppose this is not so. Then for some $\varepsilon_0 > 0$ and some k one can find an interval L of the line on which the function $q_k(t)$ has mean increment $> \lambda - \varepsilon_0/2$; but on this interval there will be nonintersecting intervals whose union has relative (on L) measure close to 1, and on each of the intervals $q_k(t)$ has mean increment $< \lambda - \varepsilon_0$. This leads to a contradiction, which proves the lemma.

With the aid of Lemma 1 the following is proved.

Lemma 2. Let the vector-functions

$$q_k(t) = \{q_k^{(1)}(t), \dots, q_k^{(n)}(t)\} \qquad (k = 1, 2, \dots)$$

be such that

$$\|q_k(t) - q_k(\tau)\| \leq a(t - \tau) \qquad (k = 1, 2, \dots;\ t \geq \tau \geq t_0).$$

Let $\tau_k \geq t_0$ $(k = 1, 2, \dots)$ and

$$q(t) = \{q^{(1)}(t), \dots, q^{(n)}(t)\} = \lim_{k \to \infty} [q_k(\tau_k + t) - q_k(\tau_k)]$$

(the limit is uniform on each interval for $t \geq 0$). Suppose that for any τ_k, $q^{(i)}(t) \in \bar{R}$.

Denote

$$\lambda = \overline{\lim}_{t-\tau \to +\infty} \frac{q^{(1)}(t) - q^{(1)}(\tau)}{t - \tau}.$$

Then for every $\eta > 0$ there exist numbers $\lambda_1, \dots, \lambda_n$ $(|\lambda_1 - \lambda| < \eta)$, numbers $\theta_j \geq t_0$, and indices k_j $(j = 1, 2, \dots)$ such that there exists

$$r(t) = \lim_{j \to \infty} [q_{k_j}(\theta_j + t) - q_{k_j}(\theta_j)]$$

(the limit is uniform on each interval for $t \geq 0$), and for every $i = 1, 2, \dots, n$

$$(\lambda_i - \varepsilon)(t - \tau) - d_\varepsilon \leq r^{(i)}(t) - r^{(i)}(\tau) \leq (\lambda_i + \varepsilon)(t - \tau) + d_\varepsilon$$

for every $\varepsilon > 0$, some $d_\varepsilon \geq 0$ (d_ε is a function of ε), and all $t \geq \tau \geq 0$.

We consider the systems

$$\dot{x} = A(t)x \text{ in } E^n; \qquad (\|A(t)\| \leq a;\ t \geq t_0); \qquad A(t) \in R; \tag{I}$$

$$\dot{y} = A(t)y + \varphi(y, t); \qquad \|\varphi(y, t)\| \leq g(t)\|y\| \tag{II}$$

($A(t)$ and $\varphi(y, t)$ are continuous in t and in y).

Definition 1. The **maximal exponent** of the vector-function $x(t)$ will be called

$$\bar{\lambda} = \overline{\lim}_{t-\tau \to +\infty} \frac{1}{t - \tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|}.$$

The **minimal exponent** of $x(t)$ is defined as

$$\underline{\lambda} = \lim_{t-\tau\to+\infty} \frac{1}{t-\tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|}.$$

Definition 2. A number λ will be called a **rough exponent** of the system (I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that from

$$g(t) = g_1(t) + g_2(t); \qquad g_1(t) < \delta; \qquad \int^{+\infty} g_2(\tau)\, d\tau < \infty$$

it follows that the system (II) has a generalized (i.e., ordinary or a shift of an ordinary or a limiting–see (^{11})) solution whose characteristic exponent $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$. The set of rough exponents of the system (I) will be called the **rough real spectrum** Λ_s of the system (I). *(It is convenient to replace by this definition the definition of the corresponding concept given in* (^{13})*.)*

Theorem 1. *For every generalized solution $x(t)$ of the system (I),*

$$\overline{\lambda} \in \Lambda_s \quad \text{and} \quad \underline{\lambda} \in \Lambda_s \qquad (A(t) \in R).$$

The proof is obtained by combining two results: Theorem 2 of note (^{13}) and Lemma 2 of the present note.

We now pass to the case where $A(t)$ is an almost periodic matrix for $-\infty < t < +\infty$.

Theorem 2. *Let $A(t)$ be an almost periodic matrix ($A(t) \in R$). Then there exists a Perron transformation $x = U(t)u$ reducing the system (I) to triangular form*

$$\dot{u} = \begin{pmatrix} b_{11}(t), \dots, b_{1n}(t) & & \\ 0 & \ddots & \vdots \\ & & b_{nn}(t) \end{pmatrix} u, \tag{1}$$

where the diagonal coefficients $b_{ii}(t)$ are "integrally close" to certain constants λ_i:

$$(\lambda_i - \varepsilon)(t-\tau) - d_\varepsilon \le \int_\tau^t b_{ii}(\xi)\, d\xi \le (\lambda_i + \varepsilon)(t-\tau) + d_\varepsilon, \tag{2}$$

$i = 1, 2, \dots, n$ (for every $\varepsilon > 0$, some $d_\varepsilon \ge 0$ (d_ε is a function of ε), and all $t \ge \tau$).

We outline the proof. Reduce the system (I) by a Perron transformation $U_1(t)$ to triangular form. It can be shown that the uniform continuity of $A(t)$ on the line implies that $U_1(t)$ is uniformly continuous on the line. Using this and Lemma 2, we obtain that there exists a limiting system

$$\dot{x} = A^*(t)x, \qquad \text{where} \quad A^*(t) = \lim_{t_k \to +\infty} A(t_k + t),$$

which is reduced by the Perron transformation $\widetilde{U}_1(t)$ to the form (1)–(2), with $\widetilde{U}_1(t)$ uniformly continuous on the line. Making use of the fact that

$$A(t) = \lim_{k \to +\infty} A^*(\theta_k + t),$$

we obtain that the system (I) itself is reduced by a Perron transformation to (1) –(2).

Theorem 3. *Let the system (I) have* n *distinct characteristic exponents* $\lambda_1, \lambda_2, \dots, \lambda_n$ *(*$A(t) \in R$ *is an almost periodic matrix).*

Then every fundamental matrix $X(t)$ of system (1) has the form

$$X(t) = S(t) \exp \left[\operatorname{diag} \left\{ \int_0^t p_1(\tau)\, d\tau, \dots, \int_0^t p_n(\tau)\, d\tau \right\} \right],$$

where $S(t)$ is a Lyapunov almost-periodic matrix; the function $p_i(t)$ $(i = 1, 2, \dots, n)$ is almost periodic and has mean value equal to λ_i.

Proof. By means of Theorem 2 and the methods of the papers (^{12,13}), we obtain that the hypotheses of B. F. Bylov' s theorem are satisfied (see (^{3}), Theorem 2). With the aid of this theorem we obtain our assertion.

Generalizing the arguments of B. F. Bylov (see (^{3})), by means of Theorem 2 we obtain, in the general case (i.e., when system (1) need not have n distinct characteristic exponents), the following theorem (generalizing Floquet' s theorem).

Theorem 4. Let $A(t) \in R$ be an almost-periodic matrix, and let $x_1(t), \dots, x_n(t)$ be a normal system of solutions of system (1), the characteristic exponent of the solution $x_i(t)$ being equal to λ_i.

Then there exists a Lyapunov almost-periodic transformation $x = S(t)u$ reducing system (1) to the block-diagonal form

$$\dot{u} = \begin{pmatrix} B^{(1)}(t) & 0 \\ 0 & B^{(m)}(t) \end{pmatrix} u,$$

where each system

$$\dot{v}_k = B^{(k)}(t) v_k$$

is such that the logarithmic derivative of the norm of each of its solutions is "integrally close" to one of the constants λ_i (the same one for the given k).

Remark 1. From the theorems proved it follows that, for an R-system with almost-periodic coefficients, $\Lambda_s = \Lambda^0$, where Λ^0 is the set of characteristic exponents of the ordinary solutions of the system.

Remark 2. The theorems proved give a method for computing the characteristic exponents of R-systems with almost-periodic coefficients, completely analogous to that given by Floquet' s theorem for systems with periodic coefficients.

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Note: Figure translations are in progress. See original paper for figures.