A THEOREM ON THE METRIZABILITY OF THE PREIMAGE OF A METRIC SPACE UNDER AN OPEN-CLOSED FINITE-TO-ONE MAPPING, AN EXAMPLE, AND UNSOLVED PROBLEMS

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A THEOREM ON THE METRIZABILITY OF THE PREIMAGE OF A METRIC SPACE UNDER AN OPEN-CLOSED FINITE-TO-ONE MAPPING, AN EXAMPLE, AND UNSOLVED PROBLEMS

(Presented by Academician P. S. Aleksandrov on 29 XII 1965)

It is proved that, under an open-closed finite-to-one mapping, the weight of the preimage does not exceed the weight of the image ([2])*. This was preceded by a result of V. Proizvolov: if $f : X \to Y$ is an open finite-to-one mapping of a locally bicompact space X onto a space Y of weight $\leq \tau$, then the weight of $X \leq \tau$ ([3]) (τ is an infinite cardinal number). In connection with this, two problems arose:

1. Is every space that can be mapped openly, closedly, and finite-to-one onto a metric space metrizable?
2. Let f be an open finite-to-one mapping onto a compact space. Is there a countable base in the preimage (equivalently: is the preimage metrizable)?

Below a positive answer will be given to the first question and a negative one to the second. This progress is not a completion. On the contrary, only now do we come closely to V. Proizvolov' s problem:

3. Does every bicompactum which is mapped onto a compactum by an open mapping with compact preimages of points have a countable base?
4. In my opinion, the following question, concerning the relationship between feathered spaces and metric spaces, also deserves attention: does there exist in the first class a nonmetrizable space that is mapped openly and finite-to-one onto some space of the second class**?

The solution of the main problem (1) reduces to the proof of the following two assertions:

I. If $f : X \to Y$ is an open finite-to-one mapping of an arbitrary space X onto a metric space Y, then in X there is a sequence $\varphi = \{\lambda_n\}$ of open covers λ_n, satisfying the following quasi-refinement condition:

$$\bigcap_{n=1}^{\infty} \lambda_n x = x$$

for every point $x \in X$.

II. If a paracompact p-space has a quasi-refining sequence of open covers, then it is metrizable.

The theorem from ([4]), characterizing paracompact p-spaces as perfect preimages of metric spaces, now makes it possible to answer the first question.

Proof of I. We shall construct the sequence $\{\lambda_n\}$ starting from the following standard representation of the mapping $f : X \to Y$.

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* In this paper only completely regular spaces and only continuous mappings are considered. The word "cover" denotes an ordinary open cover.

** A locally bicompact space that is mapped openly and finite-to-one onto a metric one is metrizable; this assertion is easily reduced to the above-mentioned result of V. V. Proizvolov.

Denote by Q_k the set of those points $y \in Y$ whose inverse image $f^{-1}y$ consists of exactly k points. Put $P_k = f^{-1}Q_k$, $Y_m = \bigcup_{k=1}^{m} Q_k$, $X_m = \bigcup_{k=1}^{m} P_k (= f^{-1}Y_m)$, $U_m = X \setminus X_m$, and $V_m = Y \setminus Y_m$, $k, m = 1, 2, \dots, \dots \infty$. Obviously, $U_m = f^{-1}V_m$. Since f is open and finite-to-one, it follows that: a) all X_m, Y_m are closed; b) all mappings $f_k = f|P_k$, $f_k : P_k \to Q_k$, are closed; c) each f_k is a local homeomorphism. Condition c) means that each P_k is locally metrizable. But from the perfectness of the mappings f_k it follows that all P_k are paracompact. Hence all P_k are metrizable. Further, Y_m is a set of type G_δ in Y, and since $X_m = f^{-1}Y_m$, U_m is the sum of a countable decreasing sequence of closed sets:

$$U_m = \bigcup_{n=1}^{\infty} F_m^n.$$

In each of the spaces P_k choose a metrically decreasing sequence of open (in P_k) covers: $\varphi_k = \{\gamma_k^n,\ n = 1, 2, \dots, \dots \infty\}$, $k = 1, 2, \dots, \dots \infty$. For each $G \in \gamma_k^n$, take some set $\widetilde{G}$, open in X, such that: 1) $\widetilde{G} \cap P_k = G$; 2) $\widetilde{G} \subseteq U_{k-1}$; 3) $\widetilde{G} \cap F_k^n = \Lambda$.

Put $\tilde{\gamma}_k^n = \{\widetilde{G} \mid G \in \gamma_k^n\}$ and $\lambda_n = \bigcup_{k=1}^{n} \tilde{\gamma}_k^n \cup (U_n)$. We shall show that for any point $x \in X$, $\bigcap_{n=1}^{\infty} \lambda_n x = x$. Let $x' \neq x$, $x' \in X$. We have $x \in P_{k_0} \subseteq U_{k_0-1}$, $x' \in P_{k_1} \subseteq U_{k_1-1}$, for some k_0, k_1. We want to prove that there is an n' such that $\lambda_{n'} x \not\ni x'$. Let us note at once that this relation is equivalent to the following one: $\lambda_{n'} x' \not\ni x$. We may therefore prove the first relation, assuming that $k_0 \leq k_1$. For each k, $0 < k \leq k_0 - 1$, choose a number $n(k)$ so that $F_k^{n(k)} \ni x'$; this is possible, since $\bigcup_{n=1}^{\infty} F_k^n = U_k \ni x'$ for $0 < k \leq k_0 - 1$. Finally, choose $n(k_0)$

from the condition $\gamma_{k_0}^{n(k_0)} x \not\ni x'$, if $x' \in P_{k_0}$, and from the condition $x' \in F_{k_0}^{n(k_0)}$, if $x' \in X \setminus P_{k_0} = U_{k_0} = \bigcup_{n=1}^{\infty} F_{k_0}^n$. Put $N = \max_{1 \le k \le k_0}(n(k) + k_0)$. Then

$$\lambda_N x = \bigcup_{k=1}^{N} \tilde{\gamma}_k^N x = \bigcup_{k=1}^{k_0-1} \tilde{\gamma}_k^N x \cup \tilde{\gamma}_{k_0}^N x. \tag{S}$$

By condition 3), no element of the system γ_k^N, $1 \le k \le k_0-1$, intersects $F_k^N \ni x'$. Therefore the first summand in the expression on the right of formula (S) does not contain the point x'. If $x' \notin P_{k_0}$, then the same argument shows that the second summand also does not contain the point x'. If, however, $x' \in P_{k_0}$, then the relation $\tilde{\gamma}_{k_0}^N x \not\ni x'$ follows from the equality $\tilde{\gamma}_{k_0}^N x \cap P_{k_0} = \gamma_{k_0}^N x$ (which follows from condition 1)) and from the fact that

$$\gamma_{k_0}^N x \subseteq \gamma_{k_0}^{n(k_0)} x \subseteq X \setminus x'.$$

Consequently, $\lambda_N x \not\ni x'$; since the points x, x' were chosen arbitrarily, we obtain $\bigcap_{n=1}^{\infty} \lambda_n x = x$ for any point $x \in X$. Assertion I is proved.

Proof of II. The following holds (see (4)).

Lemma. Let X be a topological space, and let $\{\mu_n\}$ be a sequence of its open covers satisfying the conditions:

a) $\bigcap_{n=1}^{\infty} \mu_n x = x$; b) for every $x \in X$ and every n_1 there is an n_2 such that $[\lambda_{n_2} x] \subseteq \lambda_{n_1} x$, and for all $n_2 > n_1$ one has $\lambda_{n_2} x \subseteq \lambda_{n_1} x$. Suppose, fur–

$\cdots bX$ is a bicompact Hausdorff extension of the space X, and $\{\eta_n\}$ is such a sequence of coverings of X by sets open in bX that: c)

$$\bigcap_{n=1}^{\infty} \eta_n x \subseteq X;$$

d) $[\eta_{n_1} x]_{bX} \subseteq \eta_{n_2} x$ for $n_1 > n_2,\ x \in X$.

Then $\{\xi_n\}$, where $\xi_n = \eta_n \cap \mu_n,\ n = 1, 2, \dots, \dots \infty$, is a refining sequence of open coverings of the space X.

To construct, for a paracompact p-space with a quasi-refining sequence of coverings $\{\lambda_n\}$, sequences $\{\mu_n\}$ and $\{\eta_n\}$ with the properties a), b) and c), d), respectively, is not difficult. Put $\mu_1 = \lambda_1$, and suppose that μ_n have already been defined for $n \le k$. By δ_k denote some covering of X by open sets each of whose closures is contained both in some element of μ_k and in some element of λ_{k+1}. As μ_{k+1} take any locally finite covering inscribed in δ_k. The sequence $\{\mu_k\}$ constructed by this rule satisfies conditions a) and b). Indeed, $\mu_n x \subseteq \lambda_n x$ for every n; therefore

$$x \in \bigcap_{n=1}^{\infty} \mu_n x \subseteq \bigcap_{n=1}^{\infty} \lambda_n x = x,$$

i.e.

$$\bigcap_{n=1}^{\infty} \mu_n x = x.$$

Further,

$$[\mu_{n+1} x] = [\{ \bigcup G \mid G \in \mu_{n+1},\ G \ni x \}] = \{ \bigcup [G] \mid G \in \mu_{n+1},\ G \ni x \} \subseteq \mu_n x.$$

Now consider any bicompact Hausdorff extension bX of the space X, and let $\varphi = \{\gamma_n\}$ be some feathering of X in bX, the elements of which form a decreasing sequence. We shall apply a procedure analogous to that carried out above.

Put $\eta_1 = \gamma_1$ and, assuming that η_k has already been defined, denote by η_{k+1} some covering of the space X by sets open in bX which cuts out on X a locally finite covering, the closures in bX of whose elements are contained both in elements of η_k and in elements of γ_k. Continue the described process to infinity. Then

$$\bigcap_{n=1}^{\infty} \eta_n x \subseteq \bigcap_{n=1}^{\infty} [\eta_n x \cap X]_{bX} = \bigcap_{n=1}^{\infty} [\eta_n \cap X]_{bX} x \subseteq \bigcap_{n=1}^{\infty} \gamma_n x \subseteq X^*.$$

Condition c) has been verified. Now

$$[\eta_{n+1} x]_{bX} = [\eta_{n+1} \cap X]_{bX} x \subseteq \eta_n x.$$

Thus d) is also fulfilled. In view of the lemma formulated at the beginning and the theorem on metrizability of paracompacts with a refining sequence of coverings ([1,5]), assertion II is proved. At the same time the following has been proved:

Theorem. *The inverse image of a metric space under an open-and-closed finite-to-one mapping is metrizable**.*

Example. We shall construct a normal finally compact space that decomposes into the sum of two spaces with a countable base and that is mapped openly and finite-to-one onto an interval.

Let ABC be an equilateral triangle and let $\{B_n\}, \{C_n\}$ be such sequences of points lying respectively on the sides AB, AC that $B_{nC}n \parallel BC$, $\rho(B_{nC}n, BC) = 1/n$. By X denote the set which is the union of the contours of the triangles

$ABC, AB_{nC}n,\ n = 1, 2, \ldots, \ldots \infty$. Define the topology in X as follows: in $X \setminus C$ it coincides with the topology induced by the surrounding plane; neighborhoods of the point C correspond to smooth curves S, joining the point D of the half-interval $[AC)$ with the point C, so that: 1) the set $S \setminus (D \cup C)$ lies inside the triangle ABC and 2) S touches the line BC at the point C***. The neighborhood $O_S C$ of the point C is that one of the two sets into which the triangle ABC is divided by the curve S, which contains the interval (DC), taken together with the point C.

* By $[\eta_n \cap X]_{bX}$ is denoted the system formed by the closures in bX of the elements of the covering cut out on X by the system η_n.

** In fact, we have just proved a more general assertion: a paracompact p-space that is mapped openly and finite-to-one onto a metric space is metrizable (see Question 4 and the example).

*** For brevity, in what follows we shall call such curves regular.

The space X has all the properties we need. To define an open finite-to-one mapping of it onto an interval presents no difficulty; it is induced, for example, by the orthogonal projection of X onto the interval AC (in the usual topology).

Obviously, X is the union of two spaces with a countable base:

$$X = (X \setminus C) \cup (C).$$

The first is simply the product of a half-interval and an interval, and the second is a point. Therefore X is finally compact (there is a countable network in X). From the normality of $X \setminus C$ it follows that, in order to prove the normality of X, it suffices to establish the regularity of X at C. The latter follows from the fact that for every proper curve S one can find a proper curve S' that lies wholly in $O_S C$. Then

$$[O_S C] = O_S C \cup S' \subset O_S C.$$

Remark 1. Well-known examples of finite-to-one mappings of nonmetrizable bicompacta onto compacta show that metrizability is not preserved when passing to the preimage, nor under closed finite-to-one mappings.

Remark 2. The most general formulation of a positive result in this circle of questions is now the following: if

$$f : X \to Y$$

is a multivalued continuous open-closed mapping under which the preimages of points are compact, while the images of points are finite sets, then from the metrizability of X there follows the metrizability of Y (for the terminology, see ([7])).

Indeed, f then decomposes into the superposition of two mappings*, the first of which is inverse to an open-closed finite-to-one single-valued mapping, and the second is a perfect single-valued mapping:

$$X \xleftarrow{f_1} Z \xrightarrow{f_2} Y,$$
$$\cap$$
$$X \times Y$$

where

$$Z = \{(x, y) \in X \times Y \mid y \in fx\},$$

and f_1, f_2 are induced by the projections of the product $X \times Y$ onto its factors. One can now apply the theorem proved above (and conclude that if X is metrizable, then Z is metrizable) and Stone' s theorem ([6]) (and conclude that in this case Y is also metrizable).

Remark 3. The unsolved problem of V. V. Proizvolov (see ([3])), the remark with which this paper begins, and the theorem proved above suggest the following hypothesis:

(Γ). *Metrizability is preserved under continuous open-closed multivalued mappings whose point-images are compacta (in particular, under mappings inverse to open perfect S-mappings).*

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CITED LITERATURE

[1] P. S. Aleksandrov, *Bull. Polish Acad. Sci., Ser. Math., Astr. et Phys.*, **8**, 135 (1960).
[2] A. V. Arhangel' skii, *DAN*, **166**, No. 6 (1966).
[3] V. V. Proizvolov, *DAN*, **166**, No. 1 (1966).
[4] A. V. Arhangel' skii, *Matem. sborn.*, **67** (109), No. 1, 55 (1965).
[5] R. H. Bing, *Canad. Math. J.*, **3**, 175 (1951).
[6] A. H. Stone, *Proc. Am. Math. Soc.*, **7**, 690 (1956).
[7] V. I. Ponomarev, *Matem. sborn.*, **51**, No. 4, 515 (1960).

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* An arbitrary perfect multivalued mapping was represented in this form by Yu. M. Smirnov.