

# ON WAVE MOTIONS OF A FLUID WITH SURFACE TENSION TAKEN INTO ACCOUNT

MATHEMATICS

1966

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**Abstract**

**Full Text**

UDC 517.9

*MATHEMATICS*

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## ON WAVE MOTIONS OF A FLUID WITH SURFACE TENSION TAKEN INTO ACCOUNT

*(Presented by Academician I. N. Vekua, 3 XII 1965)*

1. The purpose of the present note is to extend the results obtained in the work <sup>(1)</sup> to the more general case when the boundary conditions of the problem contain derivatives of the function describing the free surface.

We are concerned with the following problem. Inside the unit circle  $|z| \leq 1$ ,  $z = x + iy$ , determine a curve  $\gamma$ ,  $\rho = \rho(\sigma)$ ,  $\rho^2 = x^2 + y^2$ , so that in the doubly connected domain  $G_z$ , bounded by  $\gamma$  and the unit circumference  $\Gamma : |z| = 1$ , the following conditions are fulfilled:

1°. There exists a function  $\psi(x, y)$ , harmonic inside  $G_z$  and continuous in  $G_z + \gamma + \Gamma$ .

2°.  $\psi = 0$  on  $\Gamma$ .

3°.  $\psi = c_1$ ,  $c_1 = \text{const} \neq 0$  on  $\gamma$ .

4°.  $|\text{grad } \psi| = q[\rho(\sigma), \rho'(\sigma), \rho''(\sigma), \nu]$ , where  $q$  is a twice continuously differentiable function of all its arguments, defined with respect to  $\rho$  in the interval  $0 < \rho < 1$  and for all values of the remaining arguments, positive in its domain of definition;  $\nu$  is a certain set of numerical parameters.

A similar problem arises in the theory of capillary-gravity waves. Consider, in the plane  $\zeta = \xi + i\eta$ , plane periodic motions of a heavy fluid with surface tension taken into account. Then the function  $q$  from the analogue of condition 4° has the form (Bernoulli's law)

$$\bar{q} = \sqrt{c_0 - 2g\eta + 2\alpha \frac{d^2\eta}{d\xi^2} \left[ 1 + \left( \frac{d\eta}{d\xi} \right)^2 \right]^{-3/2}}, \quad (1)$$

where  $\eta = \eta(\xi)$  is the equation of the free surface;  $g$  is the gravitational constant;  $\alpha$  is the coefficient of surface tension (a physical constant of the fluid);  $c_0$  is

Bernoulli's constant. After passing to the  $z = x + iy$  plane, the function (1) is transformed into the following:

$$q(\rho, \rho', \rho'', \nu) \equiv l(2\pi\rho)^{-1} \sqrt{c_0 + \pi^{-1}gl \cdot \ln \rho + 4\pi\alpha\rho l^{-1}(\rho\sigma'^2 - \rho\rho'')(\rho^2 + \rho\sigma'^2)^{-3/2}}, \quad (2)$$

where  $l$  is the wavelength, and  $\nu$  in the present case is the set of parameters  $l$  and  $c_0$ .

We shall show that, also in the case under consideration, under certain assumptions on the function  $q$ , there exists a two-parameter family of solutions.

**2.** Using the notation of (1), we obtain the following system of integro-differential equations for determining  $\lambda$ ,  $\mu(s)$ ,  $\rho(s)$ , to the investigation of which the analysis of the original problem reduces:

$$A_0 \equiv \int_0^{2\pi} [\mu(s) + \ln q(\rho, \rho', \rho'', \nu)(s)] ds = 0,$$

$$A_1 \equiv \left| i\lambda \int_0^\sigma \exp\{i\sigma + S\mu(\sigma) + S_1 \ln q(\rho, \rho', \rho'', \nu)(\sigma)\} d\sigma + 1 \right|^2 - 1 = 0, \quad (3)$$

$$A_2 \equiv \rho^2(\sigma) - \left| 1 + \lambda L + i\lambda r(\lambda) \int_0^\sigma q^{-1}(\rho, \rho', \rho'', \nu)(\sigma) \times \right. \\ \left. \times \exp\{i\sigma + S_2\mu(\sigma) + S_0 \ln q(\rho, \rho', \rho'', \nu)(\sigma)\} d\sigma \right|^2 = 0.$$

The solution of this system must ensure the single-valuedness of the function

$$z = z(\tau) = \lambda \int_1^\tau \exp\{F(t)\} dt + 1, \quad (4)$$

where  $\tau$  is the auxiliary plane; in this plane the domain  $G_z$  corresponds to the annulus  $G_\tau$ , whose inner radius  $r$  is (a priori unknown), and whose outer radius is 1.

If the functions  $\mu(\sigma)$ ,  $\rho(\sigma)$ ,  $\rho'(\sigma)$ ,  $\rho''(\sigma)$  are  $2\pi$ -periodic, then the operators  $A_1, A_2$  assign to each triple  $(\lambda, \mu, \rho)$  functions  $\mu_1(\sigma)$ ,  $\mu_2(\sigma)$  having Hölder-continuous derivatives. Indeed, in this case, for example,  $f(\sigma) \equiv q[\rho(\sigma), \rho'(\sigma), \rho''(\sigma), \nu]$  is a  $2\pi$ -periodic function. Since the representation

$$S_0 f(\sigma) = \frac{ih_0}{\pi} \ln \sin \frac{\sigma}{2} + \tilde{S}_0 f(\sigma),$$

holds, where  $h_0 = f(0) - f(2\pi)$ ,  $\tilde{S}_0 f(\sigma) \in \text{Lip } \beta$ , then in the case  $h_0 = 0$ ,  $S_0 f(\sigma)$  also belongs to the Hölder space. The operators  $A_0, A_1, A_2$  take each triple  $X = (\lambda, \mu, \rho)$  into the triple  $X_1 = (\mu_0, \mu_1, \mu_2)$ , where  $\mu_0$  is a number. Denote by  $E$  the set of elements  $X$ , and, in accordance with the above, introduce on  $E$  the norm

$$\|X\|_E = |\lambda| + \|\mu\|_{\text{Lip } \beta}^0 + \|\rho\|_{\text{Lip } \beta}^0 + \|\rho'\|_{\text{Lip } \beta}^0 + \|\rho''\|_{\text{Lip } \beta}^0, \quad 0 < \beta < 1, \quad (5)$$

where  $\text{Lip}^0$  denotes the Hölder space of  $2\pi$ -periodic functions.  $E$  becomes a complete normed linear space. By  $E_1$  denote the Banach space of elements  $X_1$  with norm

$$\|X_1\|_{E_1} = |\mu_0| + \|\mu_1\|_{\text{Lip } \beta} + \|\mu_1'\|_{\text{Lip } \beta} + \|\mu_2\|_{\text{Lip } \beta} + \|\mu_2'\|_{\text{Lip } \beta}. \quad (6)$$

It can be shown that the mapping  $X_1 = \varphi(X, \nu)$  of the space  $E$  into  $E_1$  is continuous and continuously differentiable.

It is easy to find the trivial solution  $X_0$  of system (3), corresponding to  $\psi(\rho) = c_1 \ln \rho / \ln \rho_0$ , where  $\rho_0 = \text{const}$  is determined from the equality

$$\rho_0 q(\rho_0, 0, 0, \nu^0) \ln \rho_0 = -|c_1|, \quad (7)$$

so that  $X_0 = (q(\rho_0, 0, 0, \nu^0), -\ln q(\rho_0, 0, 0, \nu^0), \rho_0)$ . It will be shown that there exist solutions of the system close to  $X_0$  (in the sense of the chosen metric), in some neighborhood of the point  $\rho_0, \nu^0$ .

3. For this purpose, compute the Fréchet derivative  $\varphi'(X_0, \nu^0; X)$  of the mapping  $\varphi(X, \nu)$  at the point  $X_0$  for  $\nu = \nu^0$ , and consider the linearized equation

$$\varphi'(X_0, \nu^0; X) = X_1, \quad X = (\Delta\lambda, h, l) \in E, \quad X_1 = (\mu_0, \mu_1, \mu_2) \in E_1. \quad (8)$$

Equation (8) is equivalent to a certain linear boundary-value problem for the function  $\Phi_0(\tau)$ , analytic in the annulus  $G_r$ . Expanding  $\Phi_0(\tau)$  in a Laurent series, problem (8) leads to an infinite system of linear equations for the expansion coefficients, and this system is solved rather simply. The sought functions  $h(\sigma)$  and  $l(\sigma)$  are related to  $\Phi_0(\tau)$  by the relations

$$\begin{aligned} h(\sigma) &= \text{Re } \Phi_0(e^{i\sigma}), \quad l'' + q_{\rho'}(\rho_0, \nu^0) q_{\rho'}^{-1}(\rho_0, \nu^0) l' + q_{\rho}(\rho_0, \nu^0) q_{\rho}^{-1}(\rho_0, \nu^0) l = \\ &= -q(\rho_0, \nu^0) q_{\rho}^{-1}(\rho_0, \nu^0) [\text{Re } \Phi_0(\rho e^{i\sigma}) + \mu_0]. \end{aligned} \quad (9)$$

The arbitrary constants that arise in solving the differential equation in (9) are uniquely determined from the conditions of  $2\pi$ -periodicity of the functions  $l(\sigma)$  and  $l'(\sigma)$ . In what follows we consider the case  $q_{\rho'}(\rho_0, 0, 0, \nu^0) = 0$ , which occurs in applications. The case  $q_{\rho'}(\rho_0, 0, 0, \nu^0) \neq 0$  is treated analogously.

If, as we have assumed, the right-hand side of (8) belongs to the space  $E_1$ , then it can be shown that the first two derivatives and the function  $l(\sigma)$  itself are Hölder continuous and satisfy the condition of  $2\pi$ -periodicity. However, in order that the condition of  $2\pi$ -periodicity of the function  $h(\sigma)$  be fulfilled, it is necessary and sufficient that  $X_1$  satisfy the condition

$$\eta(\rho_0)\mu_0 - F(\mu_1, \mu_2) = 0, \quad \eta(\rho_0) = \frac{\pi}{12} \left[ 1 - 24 \sum_{n=1}^{\infty} \frac{\rho_0^{2n}}{(1 - \rho_0^{2n})^2} \right], \quad (10)$$

where  $F(\mu_1, \mu_2)$  is a linear functional. If, in addition, the conditions

$$\frac{q\rho_0^n}{q_\rho - n^2 q_{\rho''}} + \frac{n-1}{n+1} \frac{q\rho_0^{-n}}{q_\rho - n^2 q_{\rho''}} - \frac{\rho_0^{-n+1}}{n+1} + \frac{\rho_0^{n+1}}{n+1} \neq 0, \quad n = 1, 2, \dots, \quad (11)$$

$$\frac{q}{q_\rho} + \rho_0 - \frac{\rho_0}{1 + \ln \rho_0} \neq 0, \quad \rho_0 \neq e^{-1}, \quad q_{\rho''} \neq 0, \quad q_\rho \neq 0, \quad q_\rho q_{\rho''}^{-1} \neq n^2,$$

are satisfied, where the values of the partial derivatives and the function  $q(\rho, \rho', \rho'', \nu)$  itself are computed at the point  $(\rho_0, 0, 0, \nu^0)$ , then equation (8) is uniquely solvable. In the case  $q_\rho q_{\rho''}^{-1} = n^2$ , the corresponding condition from (11) is replaced by the condition

$$\rho_0^n + \frac{n-1}{n+1} \rho_0^{-n} \neq 0, \quad (12)$$

which always holds for  $\rho_0 > 0$ .

Under these same conditions one can show the uniqueness of the trivial solution in some small neighborhood of  $\rho(s) = \rho_0$  for  $\nu = \nu^0$ . Condition (11) for the function  $q(\rho, \rho', \rho'', \nu)$  of the form (2) can be rewritten in terms of critical mean velocities. If  $h$  denotes the mean depth of the fluid, then conditions (11) are violated for

$$c_{\text{cr.0}}^2 = gh, \quad c_{\text{cr.n}}^2 = \left[ \frac{gl}{2\pi n} + \frac{2\pi n\alpha}{l} \right] \text{th} \frac{2\pi h}{l} n. \quad (11')$$

4. From consideration of the linearized problem it follows that nontrivial solutions may arise when conditions (11) fail. However, it can be shown that nontrivial solutions also exist when these conditions are satisfied, if the parameters  $\nu_1, \nu_2, \dots$  are regarded as variable in some (generally speaking, small) neighborhood of the point  $\nu = \nu^0(\nu_1^0, \nu_2^0, \dots)$ . Using the implicit-function theorem in functional spaces, one can prove the validity of Theorem 3 from [1] under more general conditions on the function  $q$ .

**Theorem.** Let the function  $q$  depend on  $\rho(\sigma)$  and its first two derivatives, and also on  $n$  (essential) parameters. If at the point  $(\rho_0, \nu^0)$  the conditions (11) are satisfied and  $q_{\nu_i} \neq 0$ ,  $\eta(\rho_0) \neq 0$ , then in some neighborhood of the point  $\rho = \rho_0$  there exists an  $(n - 1)$ -parameter family of solutions distinct from the trivial solution  $\rho = \rho_0$ .

Thus it has been proved that the solution of the problem of capillary-gravity waves is a two-parameter family of periodic waves.

In conclusion, the author expresses deep gratitude to I. I. Danilyuk for posing the problem and for a number of valuable suggestions.

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Received  
22 XI 1965

## CITED LITERATURE

1. I. I. Danilyuk, *DAN*, **162**, No. 5, 979 (1965).

*Note: Figure translations are in progress. See original paper for figures.*

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