ON SPACES COABSOLUTE WITH METRIC SPACES

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Abstract

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ON SPACES COABSOLUTE WITH METRIC SPACES

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Two topological spaces will be called **coabsolute** if their absolutes (see ($^{14, 8–10, 13}$)) are homeomorphic to one another. As was proved in ($^{14, 8–10, 13}$), the coabsoluteness of two spaces is necessary and sufficient for the existence of an irreducible, perfect (in general, many-valued) mapping of one of these spaces onto the other.

Definition 1. A system Σ of open sets of a space X is called **dense** in this space if every open $H \subseteq X$ contains some $U \in \Sigma$. The least cardinality of a dense system in X will be called the π**-weight of the space** X.

Main theorem. *In order that a space X admit a perfect irreducible single-valued mapping onto some metrizable space, it is necessary and sufficient that the space X be a paracompact p-space* * and that in X there exist a dense system Σ of open sets decomposing into a countable sum*

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$$

of disjoint systems Σ_i, each of which is locally finite in X. If a paracompact p-space X admits a many-valued perfect irreducible mapping onto some metrizable space, then there exists a metrizable space Y (in general, another one) and a single-valued perfect irreducible mapping $f : X \to Y$.

For the proof of this theorem we shall need some auxiliary notions and propositions, which are also of independent interest.

1. Product of mappings. Suppose mappings $f_\alpha : X \to Y_\alpha$ are given from one and the same space X onto spaces Y_α. The mapping

$$f : X \to Y \subseteq \prod_\alpha Y_\alpha,$$

which assigns to each point $x \in X$ the point $\{f_\alpha x\}$ of the product $\prod_\alpha Y_\alpha$, is called the **product** of the mappings $f_\alpha : X \to Y_\alpha$. By P_f we shall denote the collection of all distinguished subsets ** of the space X under the mapping $f : X \to Y$. The following assertions hold:

Lemma 1. *If $f : X \to Y$ is the product of mappings $f_\alpha : X \to Y_\alpha$, then $f^{-1}fx = \bigcap_\alpha f_\alpha^{-1} f_\alpha x$ for all $x \in X$ and $P_f \supseteq \bigcup_\alpha P_{f_\alpha}$.*

Lemma 2. *Let ω be some open covering of the space X,*

and among the mappings $f_\alpha : X \to Y_\alpha$ there is at least one ω-mapping with respect to this ω. Then the product $f : X \to Y \subseteq \prod_\alpha Y_\alpha$ will necessarily be an ω-mapping with respect to the covering ω.

Lemma 3. Suppose that among the mappings $f_\alpha : X \to Y_\alpha$ there is at least one perfect mapping. Then the product $f : X \to Y$ of the mappings f_α is also perfect.

Lemma 4*. Let X be a paracompact p-space, and let ω be an arbitrary open covering of it. Then there exists a metrizable space Y_ω and a perfect ω-mapping $f_\omega : X \to Y_\omega$.

Lemma 5. Let X be a paracompact p-space, and let $U \subseteq X$ be an open subset of type F_σ. Then there exists a metrizable space Y_U and a perfect mapping $f_U : X \to Y_U$, under which the set U is distinguished, i.e. $U = f^{-1}fU$.

n. 2. Dense systems of open sets.

Lemma 6. Let $f : X \to Y$ be a one-to-one perfect irreducible mapping, and let $\Sigma = \{U\}$ be a system of open sets dense in X. Then the system** $f^{\#}\Sigma = \{f^{\#}U\}$ of open sets in Y is dense in Y.

Lemma 7. Let $f : X \to Y$ be a one-to-one perfect irreducible mapping, and let $\Sigma = \{V\}$ be a system of open sets dense in Y. Then the system $f^{-1}\Sigma = \{f^{-1}V\}$ is dense in X.

Lemma 8. Let $f : X \to Y$ be a many-valued (in particular, one-valued) perfect irreducible mapping (see (9)). Then the π-weight of the space X is equal to the π-weight of the space Y.

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* Paracompacts that admit a perfect mapping onto a metrizable space were studied in detail by A. V. Arhangel' skii (^{1,2}) (see also (^{11})). We shall call these spaces paracompact p-spaces. In particular, in (^{1,2,11}) intrinsic criteria are given for a space X to be a paracompact p-space. In the paper (^{4}) it is proved (in our terminology) that a paracompact space complete in the sense of Čech is a paracompact p-space.

** A set $A \subseteq X$ is called distinguished under a mapping $f : X \to Y$ if $A = f^{-1}fA$.

n. 3. Locally finite systems.

Lemma 9. Let $\Sigma = \{U_\alpha\}$ be a locally finite in X system of open sets of type F_σ. Then the open set

$$\widetilde{\Sigma} = \bigcup_\alpha U_\alpha$$

—the body of the system Σ—also has type F_σ.

Lemma 10. Let Σ be a system of open sets dense in X and σ-locally finite*** (in all of X). Then for every open set $U \subseteq X$ there is an open set $U' \subseteq U$, everywhere dense in U, of type F_σ.

Lemma 11. If in the space X there is a system Σ of open sets that is dense and σ-locally finite (in all of X), then there is a dense σ-locally finite (also in all of X) system Σ' of open sets of type F_σ.

Lemma 12. Let $f : X \to Y$ be a one-to-one perfect irreducible mapping, and let $\Sigma = \{U\}$ be a locally finite in X system of open sets; then the system $f^\# \Sigma = \{f^\# U\}$ of open nonempty sets is locally finite in Y.

n. 4. Construction of mappings.

Basic Lemma 13. Let $\Sigma = \{U_\alpha\}$ be a disjoint locally finite in X system of open sets of type F_σ. Then there exists a metrizable space Y_Σ and a perfect mapping $f_\Sigma : X \to Y_\Sigma$, under which all sets $U_\alpha \in \Sigma$ are distinguished.

Proof. By Lemma 9 the open set $\widetilde{\Sigma}$ has type F_σ, and the closed set $\Phi = X \setminus \widetilde{\Sigma}$ has type G_δ. Let

$$\Phi = \bigcap_{i=1}^{\infty} \Gamma_i.$$

By Ω_i denote the covering of the whole space X, consisting of all

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* The proof of this lemma is contained in (³).

** In order that a one-to-one mapping $f : X \to Y$ be simultaneously closed and irreducible, it is necessary and sufficient that for every open $U \subseteq X$ the set

$$f^\# U = \mathcal{E}(y \in Y,\ f^{-1} y \subseteq U)$$

be nonempty and open in Y (proved in (⁹, ¹²)).

*** A system Σ is called σ-locally finite in X if

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i,$$

where the Σ_i are locally finite in X. The systems Σ_i need not be coverings of the whole space X.

sets $U_\alpha \in \Sigma$ and also the sets Γ_i. By Lemma 4 there exists a metrizable space Y_i and a perfect Ω_i-map $f_i : X \to Y_i$.

Consider the product $f : X \to Y \subseteq \prod_{i=1}^{\infty} Y_i$ of the mappings $f_i : X \to Y_i$. By Lemma 2 the mapping f is an Ω_i-map for every i, and by Lemma 3 it is perfect. It remains now to prove that every $U_\alpha \in \Sigma$ is distinguished for the mapping f. Let $x_0 \in U_\alpha$ be arbitrary. We must prove that $f^{-1}fx_0 \subseteq U_\alpha$. Choose i_0 such that $x_0 \in \Gamma_{i_0}$. Since f is an Ω_{i_0}-map, there exists $U \in \Omega_{i_0}$ such that $f^{-1}fx_0 \subseteq U$, and this U is necessarily one of the sets $U_\alpha \in \Sigma$ (here the disjointness of the system Σ is essential). We have $f^{-1}fx_0 \subseteq U_\alpha$, as was required to prove.

Main Lemma 14. *Let in a paracompact p-space X there be a system Σ of open sets of type F_σ, decomposing into a countable sum*

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$$

of disjoint systems Σ_i, each of which is locally finite in X. Then there exists a metrizable space Y_Σ and a perfect mapping $f_\Sigma : X \to Y_\Sigma$ under which all sets $U \in \Sigma$ are distinguished.

Proof. By Lemma 13, for each Σ_i there exists a metrizable space Y_i and a perfect mapping $f_i : X \to Y_i$, under which every set $U \in \Sigma_i$ is distinguished. Consider the product $f : X \to Y$ of the mappings $f_i : X \to Y_i$. The space Y is metrizable, and the mapping f is perfect by Lemma 3. Further, by Lemma 1, all sets $U \in \Sigma$ are distinguished. The lemma is proved.

5. Proof of the Main Theorem. a) Let X be a paracompact p-space, and let $\Sigma = \{U_\alpha\}$ be a dense system in X of open sets decomposing into a countable sum

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$$

of disjoint systems Σ_i locally finite in X. By Lemma 10, in X there exists a dense system $\Sigma' = \{U'_\alpha\}$ of open sets U'_α of type F_σ, with each U'_α everywhere dense in U_α and $U'_\alpha \subseteq U_\alpha$. Denote by Σ'_i for each $\Sigma_i = \{U^i_\lambda\} \subseteq \Sigma$ the collection of all U'^i_λ. We obtain

$$\Sigma' = \bigcup_{i=1}^{\infty} \Sigma'_i,$$

and each system Σ'_i is disjoint and locally finite in X. By Lemma 14, consider a metrizable space Y and a perfect mapping $f : X \to Y$, under which all $U'^i_\alpha \in \Sigma'$ are distinguished. We shall prove that the mapping f is irreducible. Indeed, it is enough to prove that in every open set $\Gamma \subseteq X$ there is contained the complete preimage $f^{-1}y$ of some point $y \in Y$. But the system Σ' is dense in X. Therefore there exists $U'_\alpha \in \Sigma'$ such that $U'_\alpha \subseteq \Gamma$, and the set U'_α is distinguished for the mapping f.

b) Let the space X admit a one-to-one perfect mapping $f : X \to Y$ onto some metrizable space Y. By Bing' s metrization criterion (7), in Y there exists a σ-discrete base Σ,

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i,$$

where the Σ_i are discrete in Y. Every base of a space is necessarily a dense system. Therefore, by Lemma 7, the system $f^{-1}\Sigma$ is dense in X; moreover, the systems $f^{-1}\Sigma_i$ are disjoint and locally finite in X, and all $U \in f^{-1}\Sigma$ even have type F_σ.

c) Let the paracompact p-space X admit a many-to-one perfect irreducible mapping $f : X \to Y$ onto some metrizable space Y. Then there exist a space Z and one-to-one perfect irreducible mappings $p_X : Z \to X$ and $p_Y : Z \to Y$ such that

$$f = p_Y^{\#} p_X^{-1}$$

(see (8°)). Consider in Y a σ-discrete base $\Sigma = \{V\}$. Then, by Lemmas 6 and 7, the system $p_X^{\#} p_Y^{-1}\Sigma$ will be a dense system of open sets of the space X; moreover, if

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i,$$

Σ_i are discrete, then $p_X^{\#} p_Y^{-1}\Sigma = \bigcup_{i=1}^{\infty} p_X^{\#} p_Y^{-1}\Sigma_i$, and the $p_X^{\#} p_Y^{-1}\Sigma_i$ are disjoint and locally finite in X (by Lemma 12). Thus, in the space X there exists a dense system $p_X^{\#} p_Y^{-1}\Sigma$, decomposing into a countable sum of disjoint systems locally finite in X. And then, as has already been proved, there exists a metrizable space Y' and a one-to-one perfect irreducible mapping $g : X \to Y'$. The theorem is completely proved.

§ 6. Consequences.

Theorem 2. Let X be a bicompactum. Then, in order that X admit a one-to-one irreducible mapping onto some compactum, it is necessary and sufficient that the π-weight of this bicompactum be countable. If a bicompactum X admits a many-valued irreducible mapping onto some compactum, then it also admits a one-to-one irreducible mapping onto some (generally speaking, different) compactum.

Theorem 3. In order that a completely normal bicompactum X admit an irreducible mapping onto a compactum, it is necessary and sufficient that it contain an everywhere dense countable set.

Theorem 4. In order that a bicompactum be coabsolute with the Cantor perfect set, it is necessary and sufficient that it have no isolated points and have countable π-weight.

Theorem 5. In order that a space X admit a one-to-one perfect irreducible mapping onto some metric space with complete metric, it is necessary and sufficient that X be a Čech-complete paracompact space and have a dense system Σ of open sets decomposing into a countable sum of disjoint and locally finite systems.

If a space X admits a many-valued perfect irreducible mapping onto some metrizable space with complete metric, then X is a Čech-complete paracompact space and there exists a one-to-one perfect irreducible mapping of it onto some metrizable space with complete metric.

R. Engelking and A. Pełczyński proved ([6]) that a dyadic bicompactum admitting a one-to-one irreducible mapping onto some compactum is metrizable. We shall supplement this result with the following propositions, which follow easily from our main theorem:

Theorem 6. If a dyadic bicompactum is coabsolute with some compactum, then it is necessarily metrizable.

Theorem 7. Let X be a dyadic bicompactum, and let X_0 be its everywhere dense subset coabsolute with some metric space Y_0. Then Y_0 and X (and, of course, $X_0 \subseteq X$) have a countable base.

From this there immediately follows a theorem of B. Efimov ([5]): all dyadic bicompact extensions of a space with a countable base are compacta.

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Note: Figure translations are in progress. See original paper for figures.