ON ONE REPRESENTATION OF THE ANOMALOUS GRAVITATIONAL FIELD

GEOPHYSICS

1966

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Fig. 1

Figure 1: Fig. 1

Abstract

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UDC 531.5

GEOPHYSICS

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ON ONE REPRESENTATION OF THE ANOMALOUS GRAVITATIONAL FIELD

(Presented by Academician A. A. Mikhailov on 8 I 1966)

In papers ([1], [2]) the disturbing potential T of the real Earth is considered as the sum of two parts T' and δT.* The first of them, at any exterior point, is determined by the generalized Stokes formula ([2]), while the second is found in the form of the potential of a simple layer, and for the corresponding density $\delta\varphi$ a certain integral equation has been obtained ([2]). In order to avoid solving this equation, M. S. Molodenskii posed the following problem: to select such values of the anomalies on the sphere σ that, everywhere on the surface S of the real Earth G, they would give prescribed anomalous values of the field. Is it not possible (for example, by trial or by developing general methods of analytic continuation) to select such values of the anomalies on the sphere that everywhere on S the residual anomalies δg become equal to zero? In other words, is it possible to construct on the surface of the sphere such a system of anomalies which, together with the field of the normal Earth, would determine the external gravitational field coinciding on the surface S (and therefore also outside S) with the gravitational field of the real Earth? Then $\delta\varphi$, and consequently also δT, would be equal to zero. Thus the disturbing potential would be completely determined only through the anomalies selected in this way, and we would avoid the complicated process of computing the density $\delta\varphi$ of the auxiliary surface layer ([2]).

Fig. 1

In the present note the question posed by M. S. Molodenskii is answered in the affirmative. The proof given below is constructive in the sense that the proof is at the same time an algorithm for the problem being solved.

On a known surface S (it is assumed that the equation of this surface is known) of the real Earth G a function $T(S)$ is given. It is necessary to find such values $t(\tau)$ on the sphere σ that the solution of the problem

$$\Delta T' = 0 \text{ in } G_1; \qquad T'|_\sigma = t(\tau); \qquad \text{at infinity } T' = 0,$$

where G_1 is the region exterior with respect to the sphere σ, should take the prescribed values $T(S)$ on the surface S. It is natural to assume that $\sigma \in G$ (Fig. 1).

Let us prove the following proposition. For any $\varepsilon > 0$ and $T'(S) \in L_2$ there exists a bounded function $t(\varepsilon, \tau, T)$, defined on σ, such that for any point M of the region G_2 the inequality

$$|T(M) - T'(M)| < \varepsilon, \tag{1}$$

is valid, and on the surface S

$$\left\{ \iint [T(S) - T'(S)]^2 \, dS \right\}^{1/2} < \varepsilon, \tag{2}$$

where G_2 is the region exterior with respect to S, T is the disturbing poten-

* We use the notation of paper ([2]).

potential corresponding to the function $T(S)$, i.e., the solution of the problem $\Delta T = 0$ in G_1, $T|_S = T(S)$, $T = 0$ at infinity, while $T'(S)$ is the solution of the problem

$$\Delta T'(M) = 0 \text{ in } G_1; \qquad T'(M)|_\sigma = t(\varepsilon, \tau, T); \qquad T'(\infty) = 0. \tag{3}$$

Let S_1 be a closed Lyapunov surface lying inside the sphere σ and not touching it. Consider the system of functions $\left\{ \dfrac{1}{r(M_i, M)} \right\}$, where $M \in S$; the points $M_i \in S_1$ are elements of a countable set of points distributed everywhere densely on the surface S_1; $r(M_i, M)$ is the distance between the points M_i and M. The linear independence and completeness of this system on the surface S are proved in ([3,4]).

We give one more proof of the completeness of this system. Let $T(S) \in L_2$ be orthogonal to all functions of the system $\left\{ \dfrac{1}{r(M_i, M)} \right\}$. We shall show that then $T(S) = 0$. Consider the function

$$\iint_S T(S) \frac{1}{r(M_i, S)} \, dS, \qquad T(S) \in L_2, \quad M_i \in S_1. \tag{4}$$

This continuous function assumes zero values on the everywhere dense set of points $M_i \in S_1$; therefore,

$$\iint_S T(S)\frac{1}{r(M,S)}\,dS = 0, \qquad M \in S_1. \tag{5}$$

But since the potential of the simple layer (4) is harmonic inside S_1, it, by virtue of (5), is identically equal to zero also inside S_1, and consequently (5), the density $T(S) \in L_2$ of the simple layer is equal to zero. We have obtained that for any $\varepsilon > 0$ there exists such an N that the linear combination

$$\varphi(\varepsilon, S, T) = \sum_{i=1}^{N} a_i \frac{1}{r(M_i, S)} \tag{6}$$

approximates $T(S)$ in the sense of the L_2 metric with accuracy ε,

$$\left\{\iint_S [T(S) - \varphi(\varepsilon, S)]^2\,dS\right\}^{1/2} < \varepsilon. \tag{7}$$

If the system $\left\{\dfrac{1}{r(M_i, S)}\right\}$ is first orthonormalized, then as the coefficients a_i one may take the Fourier coefficients of the function $T(S)$ with respect to the orthonormalized system $\{\psi_i(S)\}$,

$$a_i = \iint_S T(S)\psi_i(S)\,dS,$$

where

$$\psi_i(S) = \sum_{k=1}^{i} A_{k,i}\frac{1}{r(M_i, S)};$$

$A_{k,i}$ are the orthonormalization coefficients. It is known that in this case (7) will be satisfied. Consider on the sphere σ the bounded function (boundedness follows from the fact that the surfaces σ and S_2 do not touch)

$$t(\varepsilon, \tau, T) = \left(\sum_{i=1}^{N} a_i \frac{1}{r(M_i, M)}\right)\Bigg|_{\sigma} \tag{8}$$

and solve for this function the exterior Dirichlet problem (3). By virtue of the uniqueness of the solution of this problem we shall have

$$T'(M) = \sum_{i=1}^{N} a_i \frac{1}{r(M_i, M)},$$

and (2) is proved.

To prove (1), consider in G_2 the difference $r(M) = T(M) - T'(M)$. As the difference of two harmonic functions, $r(M)$ is the solution of the problem

$$\Delta r(M) = 0 \text{ in } G_1; \qquad r(M)\big|_S = T(S) - T'(S); \qquad r(\infty) = 0.$$

Represent the solution of this problem by means of the Green function

$$r(M) = \iint_S \frac{\partial G}{\partial n} \left[T(S) - T'(S)\right] dS. \tag{9}$$

Applying Schwarz' s inequality to (9) and taking into account the finiteness, for any point $M \in G_2$, $M \in S$, of the integral

$$\iint_S \left[\frac{\partial G}{\partial n}\right]^2 dS,$$

we obtain (1).

Thus, we have obtained that if on the known surface S the boundary values $T(S)$ of the disturbing potential T are known, then it is possible to choose a function (6) such that it will approximate $T(S)$ in the sense of the metric L_2, and at any point M outside S the expression $\sum_{i=1}^{N} a_i \dfrac{1}{r(M_i, M)}$ will coincide, to any desired degree of accuracy, with the disturbing potential.

As the coefficients a_i, one may take the Fourier coefficients with respect to the orthonormalized system $\psi_i(S)$ (from the linear independence of the system $\left\{\dfrac{1}{r(M_i, S)}\right\}$ it follows that orthonormalization is possible). Therefore there is no need to compute the function $t(\varepsilon, \tau, T)$ on the sphere σ, although, as a check, one may compute the expansion of the function $t(\varepsilon, \tau, T)$, defined by relation (8), in spherical functions and verify the presence of spherical harmonics of the first order (as shown in (2), the amplitude of these harmonics must be negligibly small).

In conclusion, it should be noted that the main difficulties in investigations in the theory of the figure of the Earth arise in determining the surface of the real Earth S and the boundary values of the disturbing potential T, since g is not a harmonic function and T is not measured directly. Therefore, as was correctly noted in (2), p. 5, the problem of determining the figure of the geoid may be regarded as a boundary-value problem only in the first approximation.

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Received
28 XII 1965

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Note: Figure translations are in progress. See original paper for figures.