AUTOMORPHISMS OF ALGEBRAIC GROUPS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.6

MATHEMATICS

V. P. PLATONOV

AUTOMORPHISMS OF ALGEBRAIC GROUPS

(Presented by Academician A. I. Mal̆ cev on 11 X 1965)

Following Borel—Serre—Mostow, we shall call a solvable linear group G **supersolvable**, or an MP-group, if G/G_0 (G_0 is the connected component of the identity in the Zariski topology) is a finite supersolvable group, i.e. has an invariant series with cyclic factors.

First in (^{1}) (see also (^{3}), Ch. 20) for compact Lie algebras, and then in (^{2}) for finite-dimensional Lie algebras over an arbitrary field of characteristic zero, it was proved that a supersolvable group of semisimple automorphisms of a Lie algebra has an invariant Cartan subalgebra. For Lie algebras over an arbitrary field the question remains open. Apparently, its solution is connected with the solution of the well-known problem on Lie algebras with a regular automorphism (see (^{4})).

In what follows, by an algebraic group over a field Q we mean a subgroup of $GL(n, Q)$ closed in the Zariski topology.

One of the main results of this paper consists in establishing a global analogue of the Borel—Mostow theorem for algebraic groups over an algebraically closed field.

Theorem 1. *A supersolvable subgroup of semisimple matrices of an algebraic group G belongs to the normalizer of some Cartan subgroup of the group G_0.*

Theorem 1 is derived from the following stronger theorem:

Theorem 2 (first invariance theorem). *A finite supersolvable group Φ of rational automorphisms of a connected algebraic group G has an invariant Cartan subgroup, provided that the order of Φ is not divisible by the characteristic q of the ground field Q when $q > 0$.*

Corollary 1. *A finite p-group $(p \neq q)$ of rational automorphisms of a connected algebraic group Γ has nontrivial fixed points, provided that Γ is not unipotent.*

We shall call a maximal connected unipotent subgroup of an algebraic group G a **unipotent Borel subgroup**. For the unipotent case, opposite to that considered in Theorems 1 and 2, the following assertions are valid:

Theorem 3 (second invariance theorem). *A finite q-group ($0 < q$ is the characteristic of the field) Ψ of rational automorphisms of a connected algebraic group G has an invariant unipotent Borel subgroup.*

Theorem 4. *Every unipotent subgroup of an algebraic group is contained in the normalizer of some unipotent Borel subgroup.*

Corollary 2. *A finite q-group of rational automorphisms of a connected algebraic group Γ has nontrivial fixed points, provided that Γ is not a torus.*

Corollary 3. *Every unipotent subgroup of a connected algebraic group Γ belongs to some unipotent Borel subgroup. In particular, the Sylow q-subgroups of Γ are conjugate.*

The results presented, apart from their independent significance, play an important role in solving the problem of conjugacy of Sylow p-subgroups in algebraic groups (see (^{5,6})).

The proofs of the invariance theorems rely, in addition to the basic results on algebraic groups (^{7}), on results of Thompson (^{8}) and Hertzig (^{9}) on finite and algebraic groups with a regular automorphism. Of greatest importance are the well-known results of Chevalley (^{7}) on semisimple algebraic groups.

To illustrate the method, we give a detailed outline of the proof of Theorem 2.

Lemma 1. *A finite p-group ($p \neq q$) of automorphisms of an algebraic torus has nontrivial fixed points of order p.*

Lemma 2. *Let Γ be a finite supersolvable group of order $2^\alpha 3^\beta$; let S_2, S_3 be respectively its Sylow 2- and 3-subgroups. If S_2^2 is the subgroup of S_2 generated by the squares of elements, then $H = S_2^2 S_3$ is a nilpotent invariant subgroup of Γ. If $S_2^2 \neq (e)$, then Γ has central 2-elements.*

From Theorem 6 in (^{5}) there follows directly

Theorem 5. *If Φ is a finite solvable group of rational automorphisms of a connected solvable algebraic group Δ, and $(o(\Phi), q) = 1$, then in Δ there exists a Φ-invariant maximal torus.*

Outline of the proof of Theorem 2. First of all, note that since a Cartan subgroup of the group is the centralizer of a maximal torus, it suffices to prove the existence of a Φ-invariant maximal torus.

I. Suppose first that Φ is a p-group, where p is a prime distinct from q. We apply induction on $k = o(\Phi) + \dim G$. Let, in addition, the group G be strongly semisimple, i.e. contain no nontrivial normal unipotents; let φ be a nontrivial central element of Φ, $\varphi^p = 1$;

$$X^\varphi = (x \in G \mid \varphi(x) = x).$$

By Theorem 3′ of $(^9)$, in this case $X^\varphi \neq (e)$. The set X^φ is an algebraic group in view of the rationality of φ. Since for every $f \in \Phi$, $f\varphi = \varphi f$, it follows that X^φ is invariant with respect to Φ.

We shall distinguish two cases.

$\mathbf{I}_1$. $X_0^\varphi \neq (e)$; then $\dim X_0^\varphi < \dim G$, and, by the induction hypothesis, the theorem is true for X_0^φ. Let A be a Φ-invariant maximal torus of X_0^φ. If $A \neq (e)$, then the normalizer $N_G(A) \neq G$ and $N_G(A)$ is Φ-invariant. Since A is contained in some maximal torus of the group G, the maximal tori of $N_G^0(A)$, in view of their conjugacy, are maximal in G. But $\dim N_G^0(A) < \dim G$, and therefore one can use the induction hypothesis. If, however, $A = (e)$, then we apply the following device, which may be called a normalizer lifting. Since for $A = (e)$, X_0^φ is unipotent, $X_0^\varphi \subseteq B_u$, where B_u is some unipotent Borel subgroup. In view of the nilpotency of B_u, every proper subgroup of it is different from its normalizer; moreover,

$$N_{B_u}^0(X_0^\varphi) \neq X_0^\varphi,$$

if $B_u \neq X_0^\varphi$. Indeed, if Z_i, $i = 0, 1, 2, \dots, r$, are the terms of the lower central series of the group B_u, where $Z_0 = B_u$, $Z_r = (e)$, then let Z_j be such a term with the greatest j for which

$$Z_j \cap X_0^\varphi \neq Z_j;$$

then

$$Z_j \subset N_{B_u}^0(X_0^\varphi).$$

The group $N_G(X_0^\varphi)$ is Φ-invariant. If $N_G^0(X_0^\varphi)$ contains nontrivial semisimple elements, then it also contains nontrivial tori, and we obtain the situation already considered above. In the opposite case $N_G^0(X_0^\varphi)$ is unipotent and

$$N_G^0(X_0^\varphi) \neq X_0^\varphi.$$

Since an increasing sequence of connected closed subsets of G stabilizes, by considering the sequence

$$X_0^\varphi \subset N_G^0(X_0^\varphi) \subset N_G^0(N_G^0(X_0^\varphi)) \subset \cdots,$$

at some final step we obtain a Φ-invariant torus. In fact, there can be only finitely many unipotent terms in the sequence. If U is the maximal one among them, then $N_G^0(U)$ is not unipotent.

I_2. $X_0^{\varphi} = (e)$. In this case X^{Φ} is a finite group. Let Φ_p be a maximal subgroup of Φ containing φ. Then Φ_p is invariant in Φ and $[\Phi : \Phi_p] = p$. By the induction hypothesis, the group Φ_p has an invariant maximal torus $T \subset G$. It follows from Lemma 1 that X^{Φ_p} contains nontrivial fixed points of order p. The group X^{Φ_p} is finite and Φ-invariant. The group Φ induces on X^{Φ_p} a group of automorphisms $\widetilde{\Phi}$ of order p: $\widetilde{\Phi} = \{\widetilde{\varphi}\}$, $\widetilde{\varphi}^p = 1$. The automorphism $\widetilde{\varphi}$ of the group X^{Φ_p} cannot be regular; otherwise the group X^{Φ_p} is nilpotent by Thompson' s theorem (^{8}) and has p-elements by construction; consequently, $\widetilde{\varphi}$ acts regularly on a Sylow p-subgroup of X^{Φ_p}, which is impossible. Thus $\widetilde{\varphi}$, and hence the whole group Φ, has fixed points. Let $e \neq g \in X^{\Phi}$, $g = g_s g_u$ be the Jordan decomposition. Since $\Phi(g) = g$, it follows from the uniqueness of the decomposition and the rationality of the automorphisms Φ that $\Phi(g_s) = g_s$, $\Phi(g_u) = g_u$. Since g_s is contained in some maximal torus, and g_u in a unipotent Borel subgroup, the normalizer lifting can be applied.

I_3. In the case of an arbitrary group G, denote by R its solvable radical. Since R is a Φ-invariant group, Φ induces on $G^* = G/R$ a group of rational automorphisms Φ^*. In view of the strong semisimplicity of G^*, there exists a Φ^*-invariant maximal torus $T^* \subset G^*$. If D is its natural preimage in G, then D is a Φ-invariant solvable subgroup of G, whose maximal tori are maximal in G. It remains to apply Theorem 5.

II. Now let Φ be supersolvable, $o(\Phi) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_k > p_{k-1} > \dots > p_1$. We apply the analogous induction process first to the strongly semisimple group G. If $k = 1$, then this case has already been considered above.

II_1. Suppose that $o(\Phi) = 2^{\alpha_1} 3^{\alpha_2}$. Denote by P some simple component of G. Let f be an arbitrary rational automorphism of the group P. As Chevalley showed (see (^{7}), communications 18-24), f is represented in the form $f = \varphi\omega$, where ω is inner and φ is an outer automorphism of P; moreover, $\varphi^2 = 1$ for P of type A_n, D_n, E_6, or $\varphi^3 = 1$ for P of type D_4. The group Φ has a cyclic normal divisor $\Sigma = \{\sigma\}$, $\sigma^3 = 1$.

a) $\sigma(P) \neq P$. Then $P, \sigma(P), \sigma^2(P)$ are distinct simple components, and $h = x\sigma(x)\sigma^2(x)$, where $x \in P$, is contained in X^{σ}, i.e. X^{σ} is an infinite set. This case has already been analyzed above.

b) $\sigma(P) = P$. Let S_2 be a Sylow 2-subgroup of Φ.

b_1) $S_2^2 \neq (e)$. By Lemma 2, the group Φ has such a central element δ that $\delta^2 = 1$. If $\delta(P) \neq P$, then it is easy to see that X^{δ} is infinite, and the further arguments are analogous to I. If $\delta(P) = P$, then from Chevalley' s results it follows that either σ or δ is an inner automorphism, i.e. either X^{σ} or X^{δ} is infinite. Thus we again have the already studied situation.

b_2) $S_2^2 = (e)$. Let A_3 be a Sylow 3-subgroup of Φ. Then it follows from Lemma 2 that Φ/A_3 is an elementary 2-group. In view of the semisimplicity of the group A_3, by Lemma 1 it has nontrivial fixed points. It is obvious that X^{A_3} is Φ-invariant. We may suppose that X^{A_3} is finite, since the infinite case has already been analyzed earlier. Among the subgroups of the group Φ containing A_3 and having fixed points in G, choose some maximal subgroup M. If $M = \Phi$, then the group Φ has a fixed point, and the normalizer lifting is then applied. If $M \neq \Phi$, then X^M is nilpotent by Thompson' s theorem; indeed, $\Phi = MC$, where C is a nontrivial elementary 2-group, and every automorphism of C acts regularly on X^M by the choice of M. If z is a nontrivial central element of X^M, then $c(z) = z^{-1}$ for every $c \in C$, and the normalizer lifting can be used.

II_2. Let $p_k > 3$. Then the group Φ has a cyclic normal divisor $\Sigma_k = \{\sigma_k\}$, $\sigma_k^{p_k} = 1$. If $\sigma_k(P) \neq P$, then, as above, X^{Σ_k} is infinite. If $\sigma_k(P) = P$, then σ_k is an inner automorphism of P, since $(p_k, \sigma) = 1$. Consequently, X^{Σ_k} is again infinite.

II_3. The passage from a supersolvable group G to the general case is made in the same way as in item I.

Remark. Corollary 1, generally speaking, admits no strengthening and is not true even for finite cyclic groups of automorphisms that are not p-groups. In this connection we note the erroneousness of one result of Hertzig from (^{9}). In (^{9}) it is asserted (Theorems $1'$, $3'$) that a connected algebraic group G with a rational regular automorphism φ of finite order m is unipotent when $(m, q) = 1$. We indicate a minimal counterexample ($m = 6$, $\dim G = 2$). Let Q be an algebraically closed field of characteristic $q \neq 2, 3$;

$$G = \{(\alpha, \beta, \gamma) \mid \alpha\beta\gamma = 1,\ \alpha, \beta, \gamma \in Q\}.$$

$\varphi(\alpha, \beta, \gamma) = (\gamma^{-1}, \alpha^{-1}, \beta^{-1})$. Obviously, φ is a rational automorphism and $\varphi^6 = 1$. At the same time it is easily verified that φ acts regularly on G.

For fields Q of zero characteristic the following question naturally arises: let G be a connected algebraic group over Q with a rational automorphism f, and let df be the corresponding automorphism of the Lie algebra $L(G)$ (see (^{10}), Ch. 2, § 9); how does the regularity of one of these automorphisms affect the other?

It is not hard to show that the regularity of df does not imply the regularity of f. At the same time the converse assertion is true:

Theorem 6. *A rational automorphism f of the group G has an infinite set of fixed points if and only if df is not regular on $L(G)$.*

Corollary 4. *A connected algebraic group Γ over Q possessing a rational regular automorphism is solvable.*

Thus, Hertzig' s theorem on the solvability of a connected algebraic group over an algebraically closed field with a rational regular automorphism is true for arbitrary fields of zero characteristic. The question arises of the possibility of extending it to arbitrary fields (see (^{11}), Problem 72). We show that in the general case the answer is negative.

Example. Let Q be the field of rational functions in x over the field with two elements. Consider the simple algebraic group $SL(2, Q)$. For $g \in SL(2, Q)$ define $\varphi(g) = (g')^{-1}$, $\psi(g) = aga^{-1}$, where

$$a = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that then $f = \psi\varphi$ is a rational regular automorphism of $SL(2, Q)$.

Belorussian State University
named after V. I. Lenin

Received
28 IX 1965

REFERENCES

1. A. Borel, J.-P. Serre, Comm. Math. Helv., 27, 128 (1953).
2. A. Borel, G. Mostow, Ann. Math., 61, No. 3, 389 (1955).
3. Seminar "Sophus Lie," Theory of Lie algebras. Topology of groups II, IL, 1962.
4. V. A. Krekhnin, A. I. Kostrikin, DAN, 153, No. 3, 467 (1963).
5. V. P. Platonov, DAN, 153, No. 2, 270 (1963).
6. V. P. Platonov, DAN, 160, No. 3, 541 (1965).
7. C. Chevalley, *Classification des groupes de Lie algébriques*, Séminaire École Normale, Paris, 1958.
8. J. Thompson, Proc. Nat. Acad. Sci. U.S.A., 45, 578 (1959).
9. D. Hertzig, Am. J. Math., 83, 421 (1961).
10. C. Chevalley, *Theory of Lie Groups*, Vol. 2, IL, 1958.
11. *Kourovka Notebook, Unsolved Problems in Group Theory*, Novosibirsk, 1965.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.