A MATHEMATICAL MODEL OF THE INTERMITTENCY OF A TURBULENT FLOW

1966

SovietRxiv

Abstract

Full Text

UDC 532.517.4

MATHEMATICAL PHYSICS

E. A. NOVIKOV

A MATHEMATICAL MODEL OF THE INTERMITTENCY OF A TURBULENT FLOW

(Presented by Academician M. D. Millionshchikov, 24 VI 1965)

1. The problem of turbulence can be formulated in the form of a closed equation for the characteristic functional of the velocity field ([1,2]). The study of this equation is hampered by the absence of a suitable zeroth approximation. Attempts, in an Eulerian description of the small-scale structure of turbulence, to take as a basis the classical Gaussian probability distribution, or one close to it, do not give positive results.

From the point of view of the probability distribution of the velocity field, a very characteristic feature is the intermittency of the turbulent flow. In ([3]) a picture of "pulses within pulses" was proposed, schematizing the phenomenon of intermittency, and on its basis a qualitative calculation was made of the spectrum of the fourth moment of the velocity field, which proved to be in agreement with experimental data ([4]). The present article is devoted to the construction of a mathematical model of intermittency.

2. A characteristic of intermittency may be the square of the velocity gradient (which is relatively easy to measure using the "frozen-in" hypothesis):

$$y(x) = (\partial w(x)/\partial x)^2, \qquad \langle y \rangle = 2\langle \varepsilon \rangle / 15\nu. \tag{1}$$

Here x is the coordinate along the direction of the mean flow velocity, $w(x)$ is the transverse component of the velocity, $\langle \varepsilon \rangle$ is the mean value of the dissipation of kinetic energy, and ν is the kinematic viscosity. Observations show that a region of the flow in which the function $y(x)$, averaged over this region, takes a large value (which corresponds to a large local Reynolds number ([3])), in turn contains subregions with still larger values of $y(x)$, which stand out against the background of comparatively quiet subregions. We shall assume that such a picture of "cascade intermittency" possesses a certain similarity and can be traced down to scales at which the local Reynolds numbers for regions with large values of $y(x)$ become of order unity. Such an assumption is in agreement with the ideas of Richardson ([5]) and Kolmogorov ([6]). The difference is that we concentrate attention on regions with large local Reynolds numbers and only for them require similarity under further fragmentation, while the remaining regions are regarded as a certain background (in the mathematical construction set forth below, the background consists of the "tails" of the pulses).

3. Let us consider a system of random points x_{k_1}, distributed according to the Poisson law with mean density σ. The corresponding generating functional has the form (see, for example, ([7]))

$$L_1[h(x)] \equiv \left\langle \prod_{k_1} \left[1 + h(x_{k_1})\right] \right\rangle = \exp\left\{\sigma \int h(x)\, dx\right\}. \quad (2)$$

(here and below, angle brackets denote probabilistic averaging; the integral is extended over the entire real axis; the function $h(x)$ decreases sufficiently rapidly at infinity). With each of the random points x_{k_1} we associate a system of points $x_{k_2} = x_{k_1} + \lambda_1 \theta_{k_1 l}$ $(l = 1, \dots, m)$, where $\theta_{k_1 l}$ are mutually independent random variables with the same distribution density $\rho(\theta)$. Carrying out first the averaging over the quantities $\theta_{k_1 l}$, it is not hard to calculate that

$$L_2[h(x)] \equiv \left\langle \prod_{k_2} [1 + h(x_{k_2})] \right\rangle = L_1[Q_1 h(x)], \quad (3)$$

where Q_1 is a nonlinear operator defined by the equality

$$Q_1 h(x) = \left[1 + \int h(x + \lambda_1 \theta) \rho(\theta)\, d\theta\right]^m - 1. \quad (4)$$

In passing from the s-th stage to the $(s+1)$-st, we put

$$x_{k_{s+1}} = x_{k_s} + \lambda_s \theta_{k_{sl}}\ (l = 1, \dots, m), \quad \lambda_s = \lambda_1 \beta^{s-1}, \quad 0 < \beta < m\beta < 1. \quad (5)$$

Similarity is expressed in the same number m and the same distribution law of the quantities $\theta_{k_{sl}}$, with the same scale-reduction coefficient β. We have

$$L_s[h(x)] \equiv \left\langle \prod_{k_s} [1 + h(x_{k_s})] \right\rangle = \exp\left\{\sigma \int Q_{s-1} \dots Q_1 h(x)\, dx\right\}. \quad (6)$$

The expression standing under the integral sign in (6) denotes the successive application to the function $h(x)$ of the operators Q_n, which differ from the operator (4) by replacing λ_1 by λ_n.

Define the stationary random function $y_s(x)$, consisting of a sequence of pulses with centers at the points x_{k_s}:

$$y_s(x) = \langle y\rangle \alpha^{-1}(m\beta)^{1-s} \int I_s\left(\frac{x-x'}{\lambda_s}\right) \xi_s(x')\, dx', \tag{7}$$

$$\xi_s(x) = \sum_{k_s} \delta(x - x_{k_s}), \quad \langle \xi_s(x)\rangle = \sigma m^{s-1}, \tag{8}$$

$$\alpha = \sigma\lambda_1, \quad \int I_s(\theta)\, d\theta = 1, \quad \langle y_s(x)\rangle = \langle y\rangle, \tag{9}$$

where $\xi_s(x)$ is a random density function. The characteristic functionals of the random functions $\xi_s(x)$ and $y_s(x)$ are expressed through the generating functional (6):

$$\Psi_s[q(x)] \equiv \left\langle \exp\left\{i \int q(x)\xi_s(x)\, dx\right\}\right\rangle = L_s[\exp\{iq(x)\} - 1], \tag{10}$$

$$\Phi_s[z(x)] \equiv \left\langle \exp\left\{i \int z(x)y_s(x)\, dx\right\}\right\rangle = \Psi_s\left[\sigma^{-1}m^{1-s} \int z(x + \lambda_s\theta)I_s(\theta)\, d\theta\right]. \tag{11}$$

By the statistical characteristics of a **cascade-intermittent random function** (in the present construction) we shall mean the limit to which the corresponding statistical characteristics of the random functions $y_s(x)$ tend as $s \to \infty$. In this article we confine ourselves to calculating only the principal characteristic of a cascade-intermittent function—its spectral density. However, formulas (11), (10), and (6) make it possible, in principle, to calculate moments of any order. Qualitative calculations of this kind, under a certain simplifying assumption, were carried out in ([8]).

4. From (7) it is not hard to obtain the following formula for the spectral density of the function $y_s(x)$:

$$\varphi_s(p) = \frac{\langle y\rangle^2}{2\pi\sigma} \psi_s(p) f_s(p\lambda_s), \tag{12}$$

where

$$\varphi_s(p) = \frac{1}{2\pi} \int \langle y_s'(x)y_s'(0)\rangle e^{-ipx}\, dx, \quad y_s'(x) \equiv y_s(x) - \langle y\rangle, \tag{13}$$

$$\psi_s(p) = \sigma^{-1}\nu^{2(s-1)} \int R_s(x)e^{-ipx}\, dx, \qquad R_s(x) = \langle \xi_s'(x)\xi_s'(0)\rangle,$$
$$\xi_s'(x) = \xi_s(x) - \sigma\nu^{1-s}, \tag{14}$$

$$\nu = m^{-1}, \qquad f_s(x) = \left| \int I_s(\theta) e^{ix\theta}\, d\theta \right|^2, \qquad f_s(0) = 1, \qquad f_s(x) \leqslant 1. \tag{15}$$

Keeping in mind (8) and (5), set

$$\xi_{s+1}(x) = \int \left[\sum_{k_s} \delta(x - x_{k_s} - \lambda_s \theta) \chi^i_{k_s}(\theta) \right] d\theta, \qquad \chi^i_{k_s}(\theta) = \sum_{l=1}^{m} \delta(\theta - \theta_{k_s l}). \tag{16}$$

A simple calculation gives

$$\langle \chi_{i_s}(\theta) \chi_{k_s}(\theta') \rangle = \nu^{-2} \rho(\theta) \rho(\theta') + \nu^{-1} \rho(\theta) \left[\delta(\theta' - \theta) - \rho(\theta') \right] \delta_{i_s, k_s}, \tag{17}$$

where δ_{i_s,k_s} is the Kronecker delta. Next we obtain the recurrence formula

$$\begin{aligned} R_{s+1}(x) = \nu^{-2} \iint R_s(x - \lambda_s \theta + \lambda_s \theta') \rho(\theta) \rho(\theta')\, d\theta\, d\theta' \\ + \sigma \nu^{-s} \delta(x) - \sigma \nu^{-s} \lambda_s^{-1} \int \rho(x \lambda_s^{-1} + \theta) \rho(\theta)\, d\theta. \end{aligned} \tag{18}$$

This formula can also be obtained from (10) and (6) by variational differentiation. For the spectra, taking (14) into account, the recurrence formula assumes the simpler form

$$\psi_{s+1}(p) = \psi_s(p) g(p\lambda_s) + \nu^s [1 - g(p\lambda_s)], \tag{19}$$

$$g(x) = \left| \int \rho(\theta) e^{ix\theta}\, d\theta \right|^2, \qquad g(0) = 1, \qquad g(x) \leqslant 1, \tag{20}$$

$$\psi_1(p) = 1, \qquad \psi_s(0) = 1 \qquad (s = 1, 2, \ldots). \tag{21}$$

Successively applying (19), with (5) and (21) taken into account, we have

$$\psi_{s+1}(p) = \prod_{k=1}^{s} g(p\lambda_1 \beta^{k-1}) + \sum_{k=1}^{s} \nu^k [1 - g(p\lambda_1 \beta^{k-1})] \prod_{l=k+1}^{s} g(p\lambda_1 \beta^{l-1}). \tag{22}$$

Introducing the notation

$$\eta(p\lambda_1) = \lim_{s\to\infty} \psi_s(p), \qquad G(x) = \prod_{k=1}^{\infty} g(x\beta^{k-1}), \tag{23}$$

from (22) we obtain the series

$$\eta(x) = (1-\nu)\sum_{k=0}^{\infty} \nu^k G(x\beta^k) \tag{24}$$

or the equivalent functional equation

$$\eta(x) = (1-\nu)G(x) + \nu\eta(x\beta). \tag{25}$$

We shall assume that $g(x) \to 0$ as $x \to \infty$. It can be shown that in this case $G(x)$ decreases at infinity faster than any power of x, while $\eta(x)$ has the power-law asymptotic form

$$\eta(x) = Cx^{-\ln\nu/\ln\beta} \qquad (x \gg 1), \tag{26}$$

where C is a finite positive constant*, determined by the function $g(x)$.

If we assume that the shape of the pulses does not change too strongly with increasing s, then, by virtue of (15) and (5), the limit of the last factor–

* More precisely, the coefficient of proportionality in (26) oscillates about the mean value C, being a periodic function of $\ln x$ with period equal to $\ln\beta$. Experimentally, these oscillations are apparently difficult to detect, even if one assumes that the given mathematical model describes the structure of a real turbulent flow with such a degree of detail.

coefficient on the right-hand side of (12) is equal to unity. We finally obtain

$$\varphi(p) = \lim_{s\to\infty} \varphi_s(p) = \frac{C\langle y\rangle^2}{2\pi\sigma}\,(p\lambda_1)^{-1+\mu}, \tag{27}$$

$$0 < \mu = \ln m\beta/\ln\beta < 1, \tag{28}$$

where the asymptotic form (27) does not depend on the shape of the pulses, while the specific form of the distribution density $\rho(\theta)$ affects only the value of the constant C.

In (8), for comparison, a calculation of the spectrum $\varphi(p)$ was carried out based on the assumption that the fourth moment of the velocity field is related to the second in the same way as for a Gaussian probability distribution. Such an assumption was first put forward by M. D. Millionshchikov (9) for the problem of decaying turbulence. It turned out that, when applied to the study of the small-scale structure of developed turbulence, this assumption leads to $\varphi(p) = \text{const}$ in the inertial interval, which is not consistent with experimental data. At the same time, formula (27) agrees with measurements (4, 10, 11), which give the value $\mu \approx 0.4$.

5. At sufficiently small scales, the picture of cascade intermittency described above will, naturally, be distorted owing to the influence of viscosity. Within the framework of the construction set out above, viscosity can be taken into account through a smearing of the pulse shape at sufficiently large values of s. Let us denote

$$\lim_{s\to\infty} f_s(p\lambda_1\beta^{s-1}) = F(pl_*), \qquad l_* = l_\nu \mathrm{Re}_0^{-3\mu/4(4-\mu)}, \tag{29}$$

where l_* is the inner scale indicated in (3); l_ν is the Kolmogorov inner scale (6); Re_0 is the external Reynolds number. In accordance with the definition of l_*, the function (29) is equal to unity for small values of the argument. Thus, expression (27) is valid in the range of wave numbers $\lambda_1^{-1} \ll p \ll l_*^{-1}$, while for $p\lambda_* \gtrsim 1$ it must be multiplied by the correction function (29), which will lead to a sharper decay of the spectrum. The asymptotic form in the region of large wave numbers cannot be determined from purely statistical considerations without invoking the Navier–Stokes equations. An analogous situation also arises in the Lagrangian description of a turbulent flow (12). For the case of the energy spectrum of a turbulent flow in the Eulerian description, the corresponding asymptotic form was calculated in (13) and proved to be exponential in character.

Institute of Atmospheric Physics
Academy of Sciences of the USSR

Received
7 IV 1965

REFERENCES

1. E. Hopf, *J. Rat. Mech. Anal.*, No. 1, 87 (1952).
2. E. A. Novikov, *ZhETF*, 47, issue 5 (1964).
3. E. A. Novikov, R. W. Stewart, *Izv. AN SSSR, ser. geofiz.*, No. 3 (1964).
4. A. S. Gurvich, S. L. Zubkovskii, *Izv. AN SSSR, ser. geofiz.*, No. 12 (1963).
5. L. F. Richardson, *Proc. Roy. Soc. A*, 110, No. 756, 709 (1926).
6. A. N. Kolmogorov, *DAN*, 30, No. 4 (1941).

7. R. L. Stratonovich, *Selected Problems of the Theory of Fluctuations in Radio Engineering*, Moscow, 1961.

8. E. A. Novikov, *Izv. AN SSSR, Physics of the Atmosphere and Ocean*, 1, No. 8 (1965).

9. M. D. Millionshchikov, *Izv. AN SSSR*, 32, No. 1 (1941).

10. A. S. Gurvich, S. L. Zubkovskii, *Izv. AN SSSR, Physics of the Atmosphere and Ocean*, 1, No. 8 (1965).

11. S. Pond, R. W. Stewart, *Izv. AN SSSR, Physics of the Atmosphere and Ocean*, 1, No. 9 (1965).

12. E. A. Novikov, *ZhETF*, 44, issue 6 (1963).

13. E. A. Novikov, *DAN*, 139, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.