PERTURBATION OF EIGENVALUES

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Abstract

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PERTURBATION OF EIGENVALUES AND EIGENELEMENTS OF LINEAR OPERATORS

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Let λ_0 be an eigenvalue of a closed linear unbounded operator A in a complex Banach space E with domain $D(A)$ dense in E; let $\{\varphi_i\}_1^n$ be a basis of the eigen-subspace E^n, $n \geq 1$, corresponding to λ_0. Introduce $B = A - \lambda_0 I$. Suppose that one of the following conditions holds: either 1^0 the nonhomogeneous equation $By = h$ is solvable for every $h \in E$, or 2^0 there exists a system $\{\psi_k\}_1^m$, $m \geq 1$, of linearly independent bounded linear functionals such that, for solvability of the nonhomogeneous equation, it is necessary and sufficient that $(\psi_k, h) = 0$, $k = 1, 2, \ldots, m$.

Let $\{\gamma_i\}_1^n$ be a system of bounded linear functionals in E, biorthogonal to the system $\{\varphi_i\}_1^n$, and $\{z_l\}_1^m$ a system of elements of E, biorthogonal to the system $\{\psi_k\}_1^m$. Introduce the projectors

$$P = \sum_{i=1}^{n} (\gamma_i, \cdot)\varphi_i$$

and

$$Q = \sum_{k=1}^{m} (\psi_k, \cdot) z_k.$$

The operator B maps one-to-one $E^{\infty-n} = (I - P)E \cap D(A)$ onto E in case 1^0, and onto $E_{\infty-m} = (I - Q)E$ in case 2^0; moreover, the corresponding restriction of B in both cases has a bounded inverse operator, which we denote by the same letter Γ.

Consider the perturbed linear operator $A(\varepsilon) = A + H(\varepsilon)$, where ε is a small parameter from a complex Banach space $\mathcal{E}$. Let $H(0) = 0$, the domain $D(H(\varepsilon)) \supseteq D(A)$, and let $H(\varepsilon)$ be subordinate to A in the following sense: $\Gamma H(\varepsilon)$ in case

1^0, and respectively $\Gamma(I-Q)H(\varepsilon)$ in case 2^0, are bounded linear operators depending continuously on ε ($\|\varepsilon\| \leq \rho$) in the uniform operator topology. Put further $H'(0) = H_1$. The problem of perturbation theory is to find eigenvalues $\lambda = \lambda_0 + \mu(\varepsilon)$ of the operator $A(\varepsilon)$ such that $\mu \to 0$ as $\|\varepsilon\| \to 0$, as well as eigenelements corresponding to these eigenvalues (see, for example, ([1]), where an extensive bibliography is given). To find the eigenvalues and eigenvectors y, we have the equation

$$By = -H(\varepsilon)y + \mu y. \tag{1}$$

Case 1^0. $m = 0$. Put in (1)

$$y = u + \sum_{i=1}^{n} \xi_i \varphi_i, \quad \text{where } u \in E^{\infty - n},$$

$\xi_i = (\gamma_i, y)$; we obtain

$$Bu = -H(\varepsilon)u + \mu u + \sum_{i=1}^{n} \xi_i[-H(\varepsilon)\varphi_i + \mu\varphi_i]. \tag{2}$$

Applying the operator Γ to both sides, we obtain

$$u = \sum_{i=1}^{n} \xi_i u_i(\varepsilon, \mu), \quad \text{where } \ u_i = (I + \Gamma H(\varepsilon) - \mu\Gamma)^{-1}\Gamma(-H(\varepsilon)\varphi_i + \mu\varphi_i). \tag{3}$$

Theorem 1. If $m = 0$, then every number $\lambda_0 + \mu$, where μ is sufficiently small, is an eigenvalue of the operator $A(\varepsilon)$; the corresponding eigenspace is n-dimensional, with basis elements $\varphi_i + u_i(\varepsilon, \mu)$, $i = 1, \dots, n$. As μ one may take any functional $\mu(\varepsilon)$, continuous for $\|\varepsilon\| \leq \rho_1$, such that $\mu(0) = 0$.

Case 2. $m \geq 1$. As above, we arrive at (2). Applying to it the projectors $I - Q$ and Q, we obtain the system

$$(I-Q)Bu = (I-Q)\left\{-H(\varepsilon)u + \mu u + \sum_{i=1}^{n} \xi_i[-H(\varepsilon)\varphi_i + \mu\varphi_i]\right\}, \tag{4}$$

$$0 = Q\left\{-H(\varepsilon)u + \mu u + \sum_{i=1}^{n} \xi_i[-H(\varepsilon)\varphi_i + \mu\varphi_i]\right\}.$$

From the first equation we find that $u(\varepsilon, \mu)$ is determined by formulas (3). Substituting $u(\varepsilon, \mu)$ into the second equation (4), we arrive at the numerical homogeneous linear system of m equations with n unknowns

$$\sum_{i=1}^{n}(\psi_k, \{-H(\varepsilon)+\mu I\}\{\varphi_i+u_i(\varepsilon,\mu)\})\xi_i = 0, \quad k=1,2,\dots,m. \tag{5}$$

Theorem 2. There exists $\rho_1 \leq \rho$ such that, for $\|\varepsilon\| \leq \rho_1$, the formula $\lambda(\varepsilon) = \lambda_0 + \mu(\varepsilon)$ and formulas (3) establish a one-to-one correspondence between the continuously ε-dependent eigenvalues $\lambda(\varepsilon)$, $\lambda(0)=0$, and their corresponding eigenvectors $y(\varepsilon)$, and between the solutions $\mu(\varepsilon)$, $\xi_1(\varepsilon), \dots, \xi_n(\varepsilon)$ of system (5) that are continuous in ε.

We note that for $m < n$ the result of Theorem 1 is the most characteristic one, while for $m > n$, generally speaking, $A(\varepsilon)$ has no eigenvalues tending to λ_0 as $\varepsilon \to 0$.

Everywhere below $m = n$, and $\mathcal{E}$ is the complex plane.

Lemma 1. The linear operator $\widetilde{B}$, defined for $y \in D(A)$ by the formula

$$\widetilde{B}y = By + \sum_{i=1}^{n}(\gamma_i, y)z_i,$$

has a bounded inverse operator $\widetilde{B}^{-1}$, which is an extension of the operator Γ from $E_{\infty-n}$ to E.

We shall therefore denote $\widetilde{B}^{-1} = \Gamma$. It is further assumed that $\Gamma H(\varepsilon)$ is a bounded linear operator, analytic in ε. We write (5) in the form

$$\sum_{i=1}^{n} a_{ki}(\varepsilon,\mu)\xi_i = 0, \quad k=1,2,\dots,n. \tag{6}$$

In order that system (6) have a nontrivial solution, it is necessary and sufficient that $\operatorname{Det}\|a_{ki}\| = 0$. This equation can be written in the form

$$\sum_{r+s\geq n} L_{rs}^{(n)}\varepsilon^r\mu^s = 0. \tag{7}$$

We shall call (7) the branching equation of the eigenvalue λ_0. We now apply to our problem the Newton diagram method (5), using the explicit formulas we have obtained for the coefficients $L_{rs}^{(n)}$.

Theorem 3. Let $m = n$. If all $L_{rs}^{(n)} = 0$, then the result of Theorem 1 holds. If $L_{rs}^{(n)} \neq 0$, then all eigenvalues $\lambda(\varepsilon)$ of the operator $A(\varepsilon)$, continuous in ε, such that $\lambda(0) = \lambda_0$, can be represented for $\|\varepsilon\| \leq \rho_1$ in the form of convergent series in fractional powers of ε.

For such results under stronger restrictions see ([1]).

Let C be an unbounded linear operator in E, $D(C) \supseteq D(A)$, and ΓC bounded.

Definition 1 (see (2)). We shall say that an element $f \in E^n$ has a Jordan chain of length p with respect to the operators B, C, if there exist p linearly independent elements $f^{(1)} = f, f^{(2)}, \dots, f^{(p)}$ satisfying the relations $Bf^{(k)} = Cf^{(k-1)}$, $k = 2, \dots, p$, while the equation $By = Cf^{(p)}$ is unsolvable. The length of the Jordan chain of f with respect to B, C will be denoted by $J(f, B, C)$.

Definition 2. We shall call the system $\{\varphi_i\}_1^n$, $n > 1$, C-optimal if $p_j = J(\varphi_j, B, C) < +\infty$ and

$$\operatorname{Det} \left\| (\psi_i, C(\Gamma C)^{p_j - 1} \varphi_j) \right\| \neq 0.$$

In the case $C = I$, optimality means that from the elements of the Jordan chains generated by the elements of the system $\{\varphi_i\}_1^n$, one can choose a basis in the maximal invariant subspace of the operator A corresponding to λ_0.

Theorem 4. Let $m = n = 1$, $J(\varphi_1, B, I) = p < +\infty$; then there exist exactly p, counted with multiplicity, eigenvalues of the operator $A(\varepsilon)$. If, moreover, $J(\varphi_1, B, H_1) = 1$, then all these eigenvalues are distinct and are representable in the form

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \mu_k \varepsilon^k$$

(the principal value of $\varepsilon^{1/p}$ is taken).

Theorem 5. Let $m = n = 1$, $J(\varphi_1, B, I) = p < +\infty$, $L_{11}^{(1)} \neq 0$; $J(\varphi_1, \hat{B}, \hat{H}_1) = q < +\infty$; then $p - 1$ eigenvalues of $A(\varepsilon)$ can be found in the form

$$\lambda_0 + \sum_{k=1}^{\infty} \mu_k \varepsilon^{k/(p-1)},$$

and one—in the form

$$\lambda_0 + \sum_{l=1}^{\infty} \mu_l \varepsilon^{q+l-2}.$$

Theorem 6. Let $m = n > 1$ and let $\{\varphi_i\}_1^n$ be I-optimal, $J(\varphi_i, B, I) = 1$, $i = 1, \dots, n$; then there exist n eigenvalues of the operator $A(\varepsilon)$ such that $\lambda(0) = \lambda_0$, and all of them can be expanded into convergent series in fractional powers of ε. If, moreover, $\{\varphi_i\}_1^n$ is H_1-optimal and $J(\varphi_i, B, H_1) = 1$, $i = 1, 2, \dots, n$, and all roots of the algebraic equation

$$\sum_{r+s=n} L_{rs}^{(n)} \mu_1^s = 0$$

are distinct, then $\lambda(\varepsilon)$ can be found in the form

$$\lambda_0 + \sum_{k=1}^{\infty} \mu_k \varepsilon^k,$$

where μ_1 are the indicated roots.

Let us pass to the more general case (cf. (2), addendum 1). An I-optimal system $\{\varphi_i\}_1^n$ may be regarded as arranged in the following order. Let

$$p_1 < p_2 < \dots < p_l$$

be the lengths of the Jordan chains of the elements φ_j with respect to B, I. Suppose the first q_1 elements φ_i have a Jordan chain of length p_1, the next q_2 elements have a Jordan chain of length p_2, and so on; finally, the last q_l elements have a Jordan chain of length p_l. Put

$$q_1 + \dots + q_s = \tau(s), \qquad q_1 p_1 + \dots + q_{sp} s = t(s).$$

Theorem 7. Let $\{\varphi_i\}_1^n$ be I-optimal and ordered in the manner indicated above; then there exist $t(l)$ eigenvalues $\lambda(\varepsilon)$ of the operator $A(\varepsilon)$ such that $\lambda(0) = \lambda_0$, and all of them are representable by convergent series in fractional powers of ε.

If, moreover, $\{\varphi_i\}_1^n$ is H_1-optimal, $J(\varphi_i, B, H_1) = 1$, $i = 1, \dots, n$; $L_{\tau(j),\, t(l-j)} \neq 0$, $j = 1, \dots, l$, and the following l algebraic equations have distinct roots:

$$\sum L_{ij}^{(n)} \mu_1^j = 0,$$

the summation being taken over i, j satisfying the relation

$$j - r(s) = [t(l-s) - i]/p_{l-s+1}, \qquad s = 1, 2, \dots, l.$$

Then all eigenvalues $\lambda(\varepsilon)$ of the operator $A(\varepsilon)$ such that $\lambda(0) = \lambda_0$ can be found in the form

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \mu_k \varepsilon^{k/p_s}, \qquad s = 1, 2, \dots, l.$$

Conclusions analogous to those made in Theorems 4–7 are also valid for eigenvectors.

The indicated method gives a complete solution of the problem and makes it possible to investigate each concrete case to the end. As applications, we note eigenvalue problems for matrices, Fredholm integral equations of the second kind, singular integral equations, and partial differential equations of elliptic type. We have also considered the real case.

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Note: Figure translations are in progress. See original paper for figures.