Nonlinear integrals of potential type and their properties

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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In this work, we consider a nonlinear analogue of potential-type integrals and investigate its properties. When $p = p^{-1}$, the operator A under consideration coincides with the standard potential-type operator. The paper is organized into three sections. Section 1 provides the fundamental notation and definitions used throughout the study.

§ 3

The fundamental results for nonlinear potential-type integrals are established. These results serve as the primary analytical apparatus for proving several embedding theorems within spaces characterized by mixed norms. Furthermore, they represent natural generalizations of the corresponding classical results for standard potential-type integrals.

§ 1. MAIN NOTATIONS

In the following sections, unless otherwise specified, we shall adhere to the following notation:

E_n denotes an n-dimensional Euclidean space of points $x, y, ...$; m, ν, n are natural numbers satisfying the conditions $1 \leq m < n$ and $\nu = n - m$. We denote

E_m as the m-dimensional Euclidean space of vectors $x = (x_1, \dots, x_m)$ and E_ν as the ν-dimensional Euclidean space of vectors $y = (y_1, \dots, y_\nu)$.

Let D be an m-dimensional domain in E_m, and G be a ν-dimensional domain in E_ν; in particular, D may coincide with E_m and G may coincide with E_ν. Let μ and ν be non-negative integers. We define E_m as the m-dimensional space of points x and E_ν as the ν-dimensional space of points y.

Corresponding to the numbers m and ν, the n-dimensional Euclidean space E_n can be represented as the product of subspaces in two distinct ways:

$$E_n = E_m \times E_\nu$$

or

$$E_n = E_{n_1} \times E_{n_2} \times \cdots \times E_{n_k}$$

Furthermore, we introduce the notation for points in these spaces:

$$x \in E_m, \quad y \in E_\nu, \quad z = (x, y) \in E_n$$

Similarly, we represent every vector in a consistent manner. It is evident that the decompositions $X = X_R$ and $X' = X'_R$ hold, along with the corresponding equalities.

Let $U_\delta(a)$ denote an n-dimensional ball of radius δ centered at the point $a = (a_1, \dots, a_n)$, where a_i are real positive numbers. Let n be a natural number and $\delta > 0$; we denote this as $U_\delta(a) \subset \mathbb{R}^n$.

Definition. We shall say that a function f, defined on the domain $D \subset U_\delta(0)$, satisfies the (p_1, p_2) condition if there exists a constant H such that the following inequality holds:

$$|f(X) - f(X')| \leq H \cdot \sum_{i=1}^{n} |x_i - x'_i|^{p_i}$$

where p_i are positive constants characterizing the smoothness of the function relative to each coordinate.

§ 2. AUXILIARY LEMMAS

Lemma 1. Let R be an $(n-m)$-dimensional region, which may coincide with R^m, defined such that $\int\int |f(x,y)|\, dy\, dx < \infty$. Let $I < A$ and $\lambda > 0$. Then, the following estimate holds:

$$C\left[1 + \ln\frac{1}{h}\right], \text{ if } \epsilon = 0$$

and a corresponding constant depending on h if $\epsilon < 0$.

Proof. The inequality is obvious, so we shall proceed to prove the lemma. Let $\epsilon > 0$. We select positive numbers such that $\sum_{i=1}^{n} \epsilon_i < \epsilon$. Taking the last inequality into account, and utilizing the properties of the L_p norm, we obtain the desired result. Furthermore, considering that the internal integral does not depend on the vector over which the integration is performed, and keeping the initial conditions in mind, we arrive at the conclusion:

$$\|f\|_{L_p} \leq C\|f\|_M$$

By treating the expression for p_2 in the same manner as the previous one and continuing this process, we obtain the following estimate at the m-th step. Since $p_m \geq 1$, the assertion of the lemma follows directly.

Lemma 2. If $q > p_i > 1$ for $i = 1, 2, \dots, m$, and the function $f(y, x)$ is such that $|f(y, x)| \in L_{p_1,\dots,p_m}(\Omega)$, then:

$$\|f\|_{L_q} \leq C\|f\|_{L_{p_1,\dots,p_m}}$$

Proof. Based on Lemma 1, we have:

$$\left(\int |f|^q dx\right)^{1/q} \leq \left(\int \left(\int |f|^{p_m} dx_m\right)^{q/p_m} dx'\right)^{1/q}$$

By applying Lemma 1 repeatedly and continuing this process, we arrive at the required estimate at the m-th step.

§ 3. NONLINEAR POTENTIAL INTEGRAL PROPERTIES

We now consider and estimate the expression $\|f\|_q$ based on Lemma 1. By iterating this procedure, we obtain the desired bound at the final step.

Theorem. If $P/P_i > P''/(P/P_i - P_i)$, then $A < \infty$ and $V(P) < \infty$, where c is a constant independent of f, and $r = |x - y|$. For the convenience of subsequent calculations, we introduce the following notation:

$$dx(i) = dx_1 \dots dx_i$$

$$r_s = 1 - \frac{d(x_i, s)}{d(o_t, s)} - 1, \text{ if } 2 \leq s \leq k - 1$$

Next, we transform the expression for A into a form more suitable for practical application. Note that the following lower bound for Γ holds:

$$\Gamma \geq \prod_{i=1}^{n} \Gamma_i^{1/n}$$

The proof of the theorem is divided into several primary steps. Each of these steps, in turn, consists of several sub-steps. We begin with the estimation of the expression:

$$\left(\int \dots \int |f|^{q_k} dx^{(k)}\right)^{1/q_k}$$

According to the conditions of the theorem, $q_i > q_j$ for $j = 2, \dots, m$. By sequentially applying the generalized Minkowski inequality i times to the right-hand side of the last equality, we obtain $p_i < q_i$. Taking into account the estimate and introducing the notation G, we obtain the following. On the right-hand side of the inequality, we have G; therefore, based on the properties of potential-type integrals, we have:

$$\left(\int_R \dots \int_R |f|^q \, dx_1 \dots dx_n\right)^{1/q} \leq C \|k\|_{L_p(R^n)}$$

If $L_1 \cap L_2 = \{0\}$, then by applying the generalized Minkowski inequality, we obtain the following estimate:

$$\left(\int |f(x)|^p dx\right)^{1/p} \leq C \left(\sum_{i=1}^{n} \|f_i\|_{L_{p_i}}\right)$$

If we now set $1/p = \sum_{i=1}^{n} 1/p_i - (n-1)$, and account for the cases where $m_{ij} = 0$, we obtain the final estimate for I.

We now proceed to estimate the expression p_{ltfl}. The following evaluations are carried out analogously to the previous case; therefore, we provide only the essential transformations and estimates. Consequently, by evaluating $P_{f_{t-1}}$ in a similar manner, we obtain the following at the (f_{t-1})-th step. It remains only to estimate (P_1), which will conclude the first step of the process:

$$\int f \, dy \geq \Gamma \int N^* \, d^2 u$$

The second step consists of evaluating the expression. Let us denote:

$$d_a, \text{ if } a = 1; 1 < s < m - 2$$

$$x_k, \dots, x_{b+1}; 1 \leq n < k$$

By transforming the right-hand side and taking into account that $p_i < p_b$, we obtain the following based on the generalized Minkowski inequality:

$$\int_K \left|\sum_k X_k \frac{d^k}{dx^k}\right|^2 dx \leq C \, \|f\|^2_{L^2(R^n)}$$

Continuing this process, we obtain:

$$J_{x;k-1} \leq C\left(J_{f;k-2}\right)_{L(R^{n_k})}$$

$$J_{x;k-2} \leq C\|f_{x;k-3}\|_{L(R^{n_{k-3}})}$$
$$J_3 \leq C\|f\|_{L^2(R^2)}$$

We now provide an estimate for the final expression:

$$A_2 = \left(\int |f_{k-1;1}|^2\, dx\right)^{1/2} \leq C\|f_{x;1}\|$$

Thus, $\mathcal{J} \leq c\|f\|$. From the obtained estimates, we have $\mathcal{J} \leq ....$

Theorem 3. Let $P_i \in R^n$; $p_i, p_j > q_k$, $p_i > q_i$. Let δ be an arbitrary positive number satisfying:

$$0 < \epsilon < \min\left(\frac{P_k + 1}{P_k}; \frac{P_{k+1} - 1}{P_{k+1}}; \frac{\gamma + \mu}{X_{k+1}P_{k+1}}\right)$$

We apply Hölder's inequality sequentially, beginning with the innermost integral. As a result, we obtain:

$$r|7l < Y_s)P' - |$$

We shall prove the theorem for the case where the limit equals zero. Suppose that, based on the inequality, the following estimate holds:

$$P_{\nu+1}$$

where $P_{\nu+1}$ satisfies all the conditions imposed in the proof of the preceding theorem. By setting $\epsilon = \epsilon_\nu$ and considering the limit as $\nu \to \infty$, we obtain the final result.

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Note: Figure translations are in progress. See original paper for figures.