

The Dirichlet problem for degenerate elliptic equations with nonlinear lower order terms

M. I. Aliev

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.54843>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

FEBRUARY 1966, VOLUME II, No. 5

THE DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS WITH NONLINEAR LOWER-ORDER TERMS

Let D be a finite, simply connected domain with a boundary Γ consisting of a segment $[-a, a]$ on the x -axis and an arc σ lying in the half-plane $y > 0$ and resting on the axis at points $A(-a, 0)$ and $B(a, 0)$. We consider the equation

$$y^m u_{xx} + u_{yy} + f(x, y, u, u_x, u_y) = 0 \quad (m > 0) \quad (1)$$

where f is a given function of its arguments. This equation is elliptic for $y > 0$ and undergoes parabolic degeneracy on the line $y = 0$.

In the case where f is a linear function of u, u_x, u_y , specifically

$$L(u) = y^m u_{xx} + u_{yy} + au_x + bu_y + cu = f(x, y)$$

with analytic coefficients a, b, c , M. V. Keldysh (see [?]) proved the correctness of the Dirichlet problem (or the so-called modified Dirichlet problem) depending on the exponent m and the coefficient b . Subsequent work has demonstrated the correctness of a modified Dirichlet problem in a sense first proposed in [?].

The present paper investigates the Dirichlet problem for equation (1), written in the form

$$y^m u_{xx} + u_{yy} = F(x, y, u, u_x, u_y) \quad (2)$$

within a domain D bounded by the curve σ and the segment $[-a, a]$ of the x -axis. In the following, such a domain will be referred to as a "normal domain."

The Dirichlet Problem. Find a solution $u(x, y)$ to equation (2) that is twice continuously differentiable in the domain D , continuous on the boundary Γ , and vanishes on Γ .

In the normal domain, we construct a solution $u(x, y)$ to the equation

$$Tu \equiv y^m u_{xx} + u_{yy} = F(x, y) \quad (1.1)$$

which vanishes on Γ . The solution $u(x, y)$ can be expressed using the Green's function, which in turn is explicitly represented via hypergeometric series. Indeed, the Lagrange adjoint operator to T is the operator T^* defined as...

$$T^*v = yv_{yy} + v_{xx} + (2 - k)v_y.$$

It is well known [4] that for $k < n$, the value of the integral is determined by the following expression:

$$\int_0^\infty \frac{x^k}{1 + x^n} dx = \frac{\pi}{n \sin \frac{(k+1)\pi}{n}}$$

This fundamental result in complex analysis is typically derived using the residue theorem by considering a contour in the complex plane that encloses the poles of the integrand. The poles of the function $f(z) = \frac{z^k}{1+z^n}$ are located at the roots of the equation $z^n = -1$, specifically at $z_m = e^{i\frac{\pi+2\pi m}{n}}$ for $m = 0, 1, \dots, n-1$.

The convergence of this integral requires that the exponent k satisfies specific constraints relative to n . Specifically, for the integral to converge at the lower limit (near zero), we require $k > -1$, and for convergence at the upper limit (as $x \rightarrow \infty$), we require $n - k > 1$, or $k < n - 1$. When these conditions are met, the expression provides a precise analytical solution that is frequently employed in various branches of physics and engineering.

function

$$g(x, y; \xi, \eta) \quad (1.2)$$

$$r^2 = (x - \xi)^2 + 4(y^{1/2} - \eta^{1/2})^2, \quad \bar{r}^2 = (x - \xi)^2 + 4(y^{1/2} + \eta^{1/2})^2$$

Let γ be a certain constant. The hypergeometric function is a solution to the equation $T^*v = 0$ with respect to both the variables (x, y) and (ξ, η) . Moreover, at the point $x = \xi, y = \eta > 0$, it possesses a logarithmic singularity. Along with $v_1(x, y)$, the function v_2 is also a solution to the equation $T^*v = 0$. From the identity, we conclude that the function is a solution to the equation with respect to (x, y) . The function (1.3), where $\rho = x + 4y$, serves as the Green's function for the Dirichlet problem for the equation $Tu = 0$ in a normal domain. Indeed: 1) the function satisfies the equation $Tu = 0$ with respect to (x, y) and the equation $T^*v = 0$ with respect to (ξ, η) ; 2) it has a logarithmic singularity at the point $x = \xi, y = \eta > 0$; 3) if $(x, y) \in \Gamma$ or $y = 0$, then the function vanishes. In a domain D with a piecewise smooth boundary, Green's formula holds:

$$\iint_D (vTu - uT^*v) d\xi d\eta = \int_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \quad (1.4)$$

From the domain D , we excise a sufficiently small neighborhood defined by $(x - \xi)^2 + 4(y^{1/2} - \eta^{1/2})^2 < \epsilon$ around the point (x, y) . We then apply formula

(1.4) to the remaining part of the domain, where u is a solution to equation (1.1) that vanishes on Γ , and $v(\xi, \eta) = G(x, y; \xi, \eta)$. Taking the limit as $\epsilon \rightarrow 0$ (following [4]), we obtain from this formula:

$$u(x, y) = \iint_D G(x, y; \xi, \eta) F(\xi, \eta) d\xi d\eta \quad (1.5)$$

Lemma. If $F(x, y)$ is a continuously differentiable function in D , then (1.5) is a solution to equation (1.1) that satisfies the zero boundary condition on Γ . The proof that function (1.5) satisfies equation (1.1) and vanishes on Γ follows the standard arguments in potential theory. We shall demonstrate that the function $u(x, y)$ represented by formula (1.5) vanishes at $y = 0$. Substituting the expression (1.3) for G into the integral,

THE DIRICHLET PROBLEM FOR DEGENERATE ELIPTIC EQUATIONS

We split the integral (1.5) into two parts. The integral corresponding to the second term on the right-hand side of (1.3) tends to zero. It remains to consider the integral corresponding to the first term.

$$J(x, y) = y^k \iint g(x, y; \xi, \eta) F(\xi, \eta) d\xi d\eta.$$

From this, taking into account (1) and (2), we obtain $|J| < d \frac{d}{dt} \max |F(x, y)|$. Therefore, for sufficiently small values, the following estimate holds:

$$|J| < o(1) \|y - x - k\| r^4 \frac{d}{dt} \text{ or, equivalently,}$$

$$|J| < o(1) \|r - k\|$$

From this, in turn, as a result of the substitution, we obtain:

$$|J| < o(1) \frac{a^2}{4} \int (\sqrt{y} + \sqrt{t})$$

$$2(\sqrt{y} + \sqrt{t})$$

$$O(h^{2\epsilon-3} a^2 U < 0(h) \int (h + K^2)^2 dx_j (\psi^{n+1} / \delta u - 2\pi)$$

$$2^{2(\epsilon-1)} < 0(1) h^{1-\kappa} \int (z - y)^2 dy,$$

$$O(1) h^{k-\kappa-j} \text{ for } k \neq 0,$$

$$O(1)(h \ln h + h) \text{ for } k = 0. \quad (1.6i)$$

Consequently, the integral (1.5) indeed tends toward zero. We can verify the validity of the estimates in a similar manner.

$$|K| < O(1) h^{k-\kappa},$$

$\frac{y-b}{1}$ as $\epsilon \rightarrow 0$, $O(1)(\ln \dots)$. Lemma: Let f be a continuously differentiable function with respect to its arguments, such that for $(x, y) \in \Omega$ and for all z , the condition $\frac{\partial f}{\partial z} \geq 0$ holds. Then the Dirichlet problem for equation (1) cannot have more than one solution.

I. ALIEV: If u_1 and u_2 are two solutions of equation (1), then the difference $v = u_1 - u_2$ will satisfy the equation, which can be written in the form (see [?]):

$$v_{yy} + \sum_{i=1}^n a_i(x, y)v_{x_i x_i} + \sum_{i=1}^n b_i(x, y)v_{x_i} + c(x, y)v = 0$$

$$\cdot + (1-t)u_{2x} + (1-t)u_{2y} \Big] dt = 0$$

$$y^m v_{yy} + v_{xx} + f_u v + f_{u_x} v_x + f_{u_y} v_y = 0,$$

$$g = - \int_0^1 g(x, y, tu_1 + (1-t)u_2, \dots) dt.$$

Since u is equal to zero on Γ and $\Delta u < 0$ (given $f > 0$), it follows from the maximum principle that $u > 0$ in Ω . Let us consider the boundary value problem:

$$\Delta u = \lambda f(x, y, u); \quad u|_{\Gamma} = 0, \quad (3-1)$$

where λ is a certain real parameter. Let $D = \{(x, y, u) : (x, y) \in \Omega, |u| < b\}$ be a domain in the three-dimensional space of variables x, y, u . Under the assumption ...

We adopt the notation

$$L = \max_j \int G(x, \eta; \xi) d\xi d\eta.$$

It is evident that L depends on a , such that $\lim_{a \rightarrow 0} L = 0$. Consequently, L can be made sufficiently small through an appropriate choice of a .

Theorem 1. Let f be a continuously differentiable function of its arguments, and let M be a constant bounding both f and $|f'|$ within the domain B . Under these conditions...

$$ML < b < 1. \quad (3.2)$$

Then, within the class of functions $C(D)$, problem (3.1) always possesses a unique solution. By Lemma 1, problem (3.1) is equivalent to the integral equation:

$$u(x, y) = \lambda \iint_D G(x, y; \xi, t) f(\xi, t, u) d\xi dt \quad (3.3)$$

We shall prove the existence of a solution to equation (3.3) using the method of successive approximations. For the zeroth approximation u_0 , we choose any twice continuously differentiable function that vanishes on the boundary Γ and satisfies the condition $|u_0| < b$. For the n -th approximation, we define the function:

$$u_n(x, y) = \lambda \iint_D G(x, y; \xi, t) f(\xi, t, u_{n-1}) d\xi dt \quad (3.4)$$

Since, by virtue of (3.2),

$$|u_n| < b \tag{3.5}$$

THE DIRICHLET PROBLEM FOR DEGENERATE ELIPTIC EQUATIONS

For any n , the successive approximation (3.4) does not exit the domain of definition. Denoting by M_n the maximum of $|u_{n+1} - u_n|$ in the domain D , it follows from the obvious inequality:

$$|u_{n+1} - u_n| \leq |\lambda| M \iint_D |G(x, y; \xi, t)| \cdot |u_n - u_{n-1}| d\xi dt$$

we obtain:

$$M_n \leq |\lambda| M L P_0$$

$$P_n \leq (|\lambda| M L)^n P_0$$

Due to condition (3.2), the series and, consequently, the sequence converge. The limit function satisfies the integral equation (3.3) and is therefore a solution to problem (3.1). From (3.5), we conclude that condition (3.2) ensures the uniqueness of the solution to problem (3.1). This implies, in particular, that for sufficiently small normal domains (i.e., for sufficiently small values), problem (3.1) always possesses a unique solution. If we assume the domain is small, the existence and uniqueness of the solution to problem (3.1) can be guaranteed by the smallness of the parameter.

Now, let us consider the boundary value problem:

$$u_{xx} + u_{yy} + f(x, y, u, u_x, u_y) = 0 \tag{4.1}$$

Let $B = \{(x, y) \in D, -\infty < u, u_x, u_y < \infty\}$ denote a domain in the five-dimensional space of variables (x, y, u, u_x, u_y) .

Theorem. If f is a bounded, continuously differentiable function of its arguments in domain B , then a solution to problem (3.1) always exists.

It follows from the lemma that, under the assumptions of the theorem, problem (4.1) is equivalent to the integro-differential equation:

$$u(x, y) = - \iint_D G(x, y; \xi, \tau) f(\xi, \tau, u, u_\xi, u_\tau) d\xi d\tau \tag{4.2}$$

We construct a sequence of functions as follows: let u_0 be an arbitrary twice continuously differentiable function vanishing on Γ , and define u_{n+1} by:

$$u_{n+1}(x, y) = - \iint_D G(x, y; \xi, \tau) f(\xi, \tau, u_n, \frac{\partial u_n}{\partial \xi}, \frac{\partial u_n}{\partial \tau}) d\xi d\tau \tag{4.3}$$

ALIEV $\frac{\partial G(x,y;\xi,\tau)}{\partial x} d\xi d\tau$. Let us denote by M the maximum of the following three expressions: $\iint_D |f||G| d\xi d\tau$, $\iint_D |f||G_x| d\xi d\tau$, $\iint_D |f||G_y| d\xi d\tau$.

1 for $k < 0$, $(\log y)^{-1}$ for $k = 0$.

vly)..

For $k > 0$.

By virtue of the estimates (1.6), this maximum exists. The sequences $\{y_n\}$ are uniformly bounded by the value $M = \max |f|$ within the domain. Given that the function possesses only a logarithmic singularity within the domain, it is straightforward to demonstrate the equicontinuity of these sequences based on the estimates (1.6). We may then extract a convergent subsequence. Let us consider the sequence $\{\frac{\partial \phi_n}{\partial x} - \frac{\partial \phi}{\partial x}\}$ and extract a convergent subsequence therefrom. From this resulting sequence, we again extract a convergent subsequence $\phi_n(y)$. This subsequence of $\{\phi_n\}$ converges uniformly along with its derivatives, scaled by the factor $\phi(y)$. The limit of this sequence constitutes a solution to the boundary value problem (4.1). Let D be a finite, simply connected domain within the half-strip, adjacent to the x -axis along a certain segment. We denote the boundary of the domain by Γ . Let us consider the boundary value problem.

$$Lu = tp'u_{yy} + u_{xx} + au_x + bu_y - cu = F(x, y),$$

5.1 Lemma

Let $g(x, y)$ be a twice continuously differentiable function satisfying the following conditions:

- a) $-Lg > \max |F|$ in S ,
- b) $g > \max |\phi|$ on γ .

Then for the solution $u(x, y)$ of problem (5.1), the estimate $|u| \leq g$ holds. (5.2) The lemma follows from the inequality $Lv = L(\pm u - g) = \pm F - Lg > 0$, where $v = \pm u - g$.

Dirichlet Problem for Degenerate Elliptic Equations

Since $v < 0$ on γ , it follows that $v < 0$ everywhere in S , and consequently $|u| \leq g$. Lemma 4: For the solution u of problem (5.1), the following estimate holds:

$$|u| < C \max |F| + \max |\phi|, \quad (5.3)$$

where K is a constant. The function

$$g(x, y) = \max |F| [\exp a(x^2)]$$

where $a > 0$ is a certain real number, satisfies the given condition. To ensure it also satisfies condition a), it is sufficient to require that the following inequality holds:

$$(a + a^2) \exp a(x - x_1) > 1,$$

and this is always possible through the appropriate selection of a . Using (5.2), we obtain the estimate (5.3), where $C = \exp a(x_2 - x_1)$. Similar estimates for uniformly elliptic equations can be found in [?]. Now, let us return to problem (4.1).

Theorem 3. Let f be a continuously differentiable function of its arguments in the domain D , uniformly bounded with respect to u and satisfying the condition $f_u > 0$. Then problem (4.1) always possesses a solution, and moreover, this solution is unique.

Uniqueness was already proven in Section 2. We shall prove the existence of the solution using the method proposed in [?] for the case of the Laplace operator. Problem (4.1) can be written in the equivalent form:

$$\begin{aligned} \Delta u - u &= f(x, y, u) - u, \\ u|_{\Gamma} &= \phi. \end{aligned} \tag{5.4}$$

According to the conditions of the theorem, $f_u > 0$ and, furthermore, $|f(x, y, u)| \leq M = \text{const}$. Applying the estimates (5.3) to the solution of problem (5.4), we obtain:

$$|u| \leq C \max |f(x, y, u)| \leq CM = A.$$

Let us introduce a new function $v(u)$ with the following properties: it is defined for $u \in (-\infty, \infty)$, is continuously differentiable and monotonically increasing, coincides with u for $|u| \leq A$, and satisfies the condition $|v(u)| \leq 2A$ for all other values. Consider the new boundary value problem:

$$\Delta u - u = f(x, y, v(u)) - v(u), \quad u|_{\Gamma} = \phi. \tag{5.5}$$

$$g(x, y, u, u_x, u_y) = f(x, y, v(u), u_x, u_y).$$

Problem (5.5) can also be expressed in the following form:

$$u|_{t=0} = 0, \quad 0 < x < 1. \tag{5.6}$$

Based on the properties of the function noted above, we obtain $\frac{\partial f}{\partial v} \geq \frac{\partial v}{\partial u}$. On this basis, for the solution to problem (5.6) (or (5.5)), we have from (5.3):

$$f(x, \max(y, 0), u) = f(x, y, u)$$

Thus, problems (4.1) and (5.5) coincide. Consequently, the solution to problem (5.5) is simultaneously a solution to problem (4.1). Since

$$|g(x, y, v, u_x, u_y)| < \sup |f(x, y, t, u_x, u_y)| \leq M^*,$$

$$|t| \leq 2A$$

By virtue of the theorem, problem (5.5) has a solution. **Theorem.** Suppose that in (4.1) the function is represented as $f = f(x, y, u, u_x) + f_1$, where f and f_1 are continuously differentiable functions such that $\frac{\partial f}{\partial u} > 0$, f_x is uniformly bounded with respect to u , and f_1 is uniformly bounded with respect to its arguments. Then problem (4.1) always has a solution. The proof of this theorem is carried out in the same manner as the proof of Theorem 1. Let D_δ be a domain consisting of points in the normal domain D satisfying the condition $y > \delta$, where δ is a sufficiently small positive number. **Lemma.** Let $F(x, y)$ be a continuously differentiable function satisfying the estimate

$$|F(x, y)| < y^\beta M, \tag{6.1}$$

where $\delta < \beta < 1$ and $M > 0$ are certain constants. Then formula (1.5) provides a solution to equation (1.1) that takes a zero value on the boundary. It is evident that for any $y > 0$, the function $u(x, y)$ represented by formula (1.5) is a solution to equation (1.1). The fact that the function vanishes on the boundary is proved in the same way as in Lemma 1. Let us show that $\lim_{y \rightarrow 0} u(x, y) = 0$. Substituting the expression (1.3) into (1.5) instead of F , we observe that the integral corresponding to the second term of the right-hand side of (1.3) tends to zero. Let us consider the integral corresponding to the first term: $I = \iint_D P(x, y, \xi, \eta) F(\xi, \eta) d\xi d\eta$. For sufficiently small y , we have $I = O(1) \int_0^y \eta^{\beta-1} d\eta$.

THE DIRICHLET PROBLEM FOR DEGENERATE ELIPTIC EQUATIONS

From here, as a result of the substitution $\xi = x + 2(\sqrt{y} + \sqrt{\eta})\alpha$, we obtain

$$\int_0^y \frac{\eta^\beta d\eta}{2\sqrt{y} - 2\sqrt{\eta} - 2\sqrt{y}}$$

where $\xi = y/z$.

Suppose M is some constant and δ is a sufficiently small positive number. Then we have $(y)^\beta U + \alpha y^\beta - (1 + \alpha\delta)^\beta = O(1)y^\beta + y^\beta(O(1)y^\beta \ln y)$ as $y \rightarrow \delta$. Consequently, for $\delta < 1$, the integral tends to zero, which proves the assertion of the lemma. Similarly, we obtain the estimates

$$|u_x| = O(1)y^{\beta-1/2}, \quad |u_y| = \begin{cases} O(1)y^{\beta-1} & \text{if } \beta < 1 \\ O(1) \ln y & \text{if } \beta = 1 \end{cases} \tag{6.2}$$

Theorem. Let $f(x, y, u, u_x)$ be a continuously differentiable function of its arguments in the domain $\{(x, y) \in D, |u| < \infty, |u_x| < \infty\}$. If for any (x, y) in

domain D the inequality

$$|f(x, y, u, u_x)| \leq y^{-\beta} M \quad (6.3)$$

holds for some constants $\beta < 1$ and $M > 0$, then a solution to problem (4.1) exists. Using the lemma, we reduce problem (4.1) to an equivalent equation of the form (4.2). As in the proof of the previous theorem, a sequence of functions is constructed analogous to the sequence (4.3). Let M be the maximum of the expressions: $\max |u|, \int_0^y \eta^{-\beta} d\eta, \int_0^y \eta^{-\beta} d\eta, \dots$

$$U_k(y) = y_k$$

$$U_k(y) = 1 \text{ for } \sigma = k, \text{ and } U_k(y) = 0 \text{ for } \sigma < k.$$

By virtue of the estimates in (6.2), this maximum exists. The sequences $U_k(y)$ are uniformly bounded by a constant and are equicontinuous. By applying a diagonal process,

[Figure 160: see original paper] M. I. ALIEV and colleagues (see Theorem [N]) have shown that one can extract a subsequence that converges, along with its derivatives with respect to y and its derivatives with respect to x scaled by the factor $t_y(y)$. The limiting function will be a solution to problem (4.1). We now consider the boundary value problem

$$T_m u = y^m u_{yy} + u_{xx} + ky^{m-1} u_y = F(x, y)$$

$$u_{yy} + y^m u_{xx} = 0, \quad 1 < m < 2. \quad (7.1)$$

When f does not depend on the variable, by applying the transformation

$$x = \xi, \quad z = \frac{2}{2-m} y^{\frac{2-m}{2}}$$

problem (7.1) is reduced to the problem $\Delta u = 0$. In this case, $u(\xi, z)$ is defined in the domain L , where L is the boundary of the region bounded by the curve g and a segment of the ξ -axis. If the function in (7.1) satisfies the conditions of Theorems 1-5, then Δu will also satisfy these same conditions. Furthermore, we have $\alpha < 1$. Consequently, the study of problem (7.1) for $\alpha < 1$ and $1 < m < 2$ is reduced to the case already studied above. To investigate problem (7.1) when the right-hand side depends on the variables, we construct the Green's function of the Dirichlet problem for the equation in the normal domain. The adjoint operator in the sense of Lagrange to the operator L is the operator $L^* = \frac{\partial^2}{\partial y^2} + y^m \frac{\partial^2}{\partial x^2} + (2m - k)y^{m-1} \frac{\partial}{\partial y}$.

It is known [?] that for $k < 1$, the function

$g(x, y; \xi, \eta) = q(r^2)^{-\beta} F(\beta, \beta, 2\beta; 1 - \sigma)$, where q is a certain constant and F is the hypergeometric function, is a solution of the equation $L^* u = 0$ with respect to the variables (x, y) . The function

The Dirichlet Problem for Degenerate Elliptic Equations

The function $(2 - m)$ serves as the Green's function for the Dirichlet problem of the equation in a normal domain. By applying Green's formula as discussed in Section 1, we can derive a representation for the solution to the equation $u = F(x, y)$ that vanishes on the boundary Γ . This representation is given by:

$$u(x, y) = - \iint_D G(x, y; \xi, \eta) F(\xi, \eta) d\xi d\eta, \quad (7.4)$$

where $G(x, y; \xi, \eta)$ is the Green's function defined in (7.3).

Lemma

If $F(x, y)$ is a continuously differentiable function in the domain D , then the function defined by (7.4) is a solution to the equation $u = F(x, y)$ that satisfies the zero boundary condition on Γ .

Proof. The proof is conducted analogously to the proof of the previous lemma. As $y \rightarrow 0$, the function (7.4) and its derivatives satisfy the asymptotic estimates $AO(1)$ and $O(y^m)$ respectively, ensuring the stability and existence of the solution within the specified functional space.

... when $k = m - 1$

Using the estimation lemma (7.5) and the corresponding theorem, results analogous to those previously established can be proven for the current problem.

Lemma 7.1

Let $F(x, y)$ be a continuously differentiable function in the domain that satisfies the estimate $|F(x, y)| \leq C$. Then the function defined by (7.4) is a solution to the equation $F(x, y)$. Furthermore, for the function (7.4) and its derivatives, the following estimates hold:

$$\begin{aligned} |u| &\leq C(1 + \ln y) \\ \left| \frac{\partial u}{\partial x} \right|, \left| \frac{\partial u}{\partial y} \right| &\leq Cy^{-1}(1 + \ln y) \end{aligned}$$

Specifically, for $t \in [0, \delta]$, the estimates (7.6) take the form:

$$\begin{aligned} |u| &\leq O(1)y^{m-1} \\ |u'| &\leq O(1)y^{m-2} \end{aligned} \quad (7.6)$$

This lemma is proved in a manner analogous to the previous estimation lemma, and the bounds in (7.6) are obtained similarly to the estimates in (6.2).

Theorem 6. Let f be a continuously differentiable function of its arguments in the domain B , and let there exist an estimate in B such that $|f(x, y, u, u_x, u_y)| \leq M|u|^m + N$, where $M, N > 0$ and $0 \leq m < 1$ are certain constants. Then a solution to problem (7.1) exists.

The proof of this theorem utilizes Lemma 5' and the estimates (7.5), and is carried out analogously to the proof of Theorem 5.

In conclusion, I express my gratitude to A. V. Bitsadze, Corresponding Member of the Academy of Sciences of the USSR, for the formulation of the problem and for his consultations during the execution of this work.

References

1. Keldysh, M. V. *Doklady Akademii Nauk SSSR*, 77, No. 2, 181-183, 1951.
2. Tersenov, S. A. *Doklady Akademii Nauk SSSR*, No. 4, 670-673, 1957.
3. Bitsadze, A. V. *Proceedings of the 3rd All-Union Mathematical Congress*, 3, 1957.
4. Karol, I. L. *Matematicheskii Sbornik*, 38(80), No. 3, 261-281, 1956.
5. Courant, R. *Partial Differential Equations*. Moscow, "Mir" Publishing House, 2, 1964.
6. Smirnov, M. M. *Vestnik Leningradskogo Gosudarstvennogo Universiteta (LGU)*, No. 13, Issue 3, 73-78, 1961.

Affiliation: Novosibirsk State University

Received by the Editorial Board: September [Year Omitted]

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.