ON THE STABILITY OF STATIONARY SOLUTIONS OF A MIXED PROBLEM

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Abstract

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ON THE STABILITY OF STATIONARY SOLUTIONS OF A MIXED PROBLEM

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For applications it is important to study the behavior, as $t \to \infty$, of solutions of the problem describing a chemical process inside a flat catalyst grain ([1]):

$$\frac{\partial \theta}{\partial t} = \Pi_1 \frac{\partial^2 \theta}{\partial \xi^2} + kQ_1 e^{\theta/(1+b\theta)} A(x), \tag{1}$$

$$\frac{\partial x_i}{\partial t} = \Pi_{i+1} \frac{\partial^2 x_i}{\partial \xi^2} + k e^{\theta/(1+b\theta)} A(x) \quad (i = 1, \dots, n);$$

$$\theta\big|_{t=0} = \theta_0(\xi), \qquad x_i\big|_{t=0} = \overline{x_i}(\xi),$$

$$\frac{\partial \theta}{\partial \xi}\bigg|_{\xi=0} = \frac{\partial x_i}{\partial \xi}\bigg|_{\xi=0} = \theta\big|_{\xi=1} = 0, \qquad x_i\big|_{\xi=1} = \beta_i. \tag{2}$$

Here $\Pi_i, k, Q_1, \beta_i, b$ are positive constants, $\beta_i < 1$; θ is the temperature in the catalyst grain; x_i are the concentrations of the reacting substances; $A(x)$ is a certain linear function of the variables x_i.

Consider the case $b = 0,\ n = 1,\ \Pi_1 = \Pi_2,\ \theta_0 - Q\overline{x}_1 = \beta_1,\ A(x) = 1 - x_1$. Let $u(\xi, t) = \theta(\xi, t)$ for $\xi \geq 0$, and let $u(\xi, t)$ be an even function of ξ. Then, under our assumptions, the problem reduces to the study of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + \lambda (Q - u) e^u \tag{3}$$

under the condition

$$u\big|_{t=0} = u_0(\xi), \qquad u\big|_{\xi=-1} = u\big|_{\xi=1} = 0, \tag{4}$$

where Q and λ are positive constants.

From the results of papers ([2,3]) there follows the existence and uniqueness of a twice continuously differentiable solution of problem (3)–(4) (for $-1 \le \xi \le 1$; $t \ge 0$) such that $\partial u/\partial \xi\big|_{\xi=0} = 0$. If $0 \le u_0(\xi) \le Q$, then it is easy to prove that $0 \le u(\xi,t) \le Q$ and $|\partial u/\partial \xi| < M$, where M is an absolute constant. Hence follows the compactness (in t) of the trajectory $u(\xi,t)$: for any sequence $t_1 \to \infty$ there exists a t_{i_k} such that

$$u(\xi, t_{i_k}) \xrightarrow[k\to\infty]{c} \varphi(\xi).$$

As follows from the results ([3,4]), for given values of the parameters there can exist only a finite number of stationary (independent of t) solutions (direct computations carried out by V. S. Beskov show that problem (3), (4) can have no more than three stationary solutions). Hence it follows that the function $\varphi(\xi)$ does not depend on the choice of the sequence t_k,

$$\lim_{\substack{t \\ k\to\infty}} u(\xi, t_k) = \varphi(\xi)$$

for any sequence $t_k \to \infty$, since it is not difficult to verify, using the boundedness of $u(\xi,t)$, that $\varphi(\xi)$ satisfies equation (3).

We shall call a stationary solution $\varphi(\xi)$ **stable** if one can specify $\varepsilon > 0$ such that from the condition $|u_0(\xi) - \varphi(\xi)| < \varepsilon$ it follows that $\|u(\xi,t) - \varphi(\xi)\|_{L_2} \xrightarrow[t\to\infty]{} 0$. We shall indicate a practically easily verifiable necessary and sufficient condition for the stability of a stationary solution.

Let $\varphi(\xi)$ be a solution of problem (3)–(4), independent of t, $u(\xi,t) = w + \varphi(\xi)$. Then

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \lambda(Q - \varphi - 1)e^{\varphi} w + \rho(\xi,t) w^2; \tag{5}$$

$$w\big|_{t=0} = w_0(\xi), \qquad w\big|_{\xi=-1} = w\big|_{\xi=1} = 0, \tag{6}$$

where $\rho(\xi,t)$ is a function bounded and continuous for all ξ and t. In [4, 5] it is proved that $\varphi(\xi)$ is a monotone function for $\xi \ge 0$, $0 \le \varphi < Q$, and that the solution of the equation obtained from

$$\frac{d^2 v}{d\xi^2} + \lambda(Q - \varphi - 1)e^{\varphi} v = 0 \tag{7}$$

by the substitution $\varphi(\xi) = y$ is the function

$$z(y) = \frac{1}{F'(y)} + \sqrt{F(y) - F(y_0)} \int_{y_0}^{y} \frac{F''(\eta)\, d\eta}{F'(\eta)^2 \sqrt{F(\eta) - F(y_0)}}, \tag{8}$$

where $y_0 = \varphi(0)$; $F(y) = (y - Q - 1)e^y$; any stationary solution $\varphi(\xi)$ is completely determined by the value $y_0 = \varphi(0)$, while y_0 is found from the functional equation

$$f(y_0) = \int_0^{y_0} \frac{d\eta}{\sqrt{F(\eta) - F(y_0)}} = \sqrt{2\lambda},$$

and it is easy to see that

$$df/dy_0 = k(y_0)z(0),$$

where $k(y_0)$ is a strictly negative function.

We shall prove that the necessary and sufficient condition for stability of the stationary solution $\varphi(\xi)$ is the condition $z(0) < 0$ for $y_0 = \varphi(0)$. (The only exception is the case $z(0) = 0$ for a certain completely determined Q.)

Indeed, let $z(0) < 0$; then $z(y) < 0$ for $0 \le y \le y_0 < Q$, since otherwise for some y_1 one would have $z(y_1) > 0$, $z'(y_1) = 0$. But it is easy to see that $z'(y_1) = 0$ implies $z(y_1) = 1/F'(y_1) < 0$, which leads to a contradiction. Returning to the variable ξ, we obtain the function $v(\ ) \equiv z(\varphi(\xi))$, satisfying equation (7), and, extending it in an even manner, which is possible since $v'(0) = 0$, we obtain a solution of equation (7) which does not vanish for any ξ, $-1 \le \xi \le 1$. From the comparison theorem it is now easy to conclude that all eigenvalues of the first boundary-value problem for equation (7) are negative, or, equivalently, that for any $v(\xi)$, $v(-1) = v(1) = 0$, for some $q > 0$ the inequality

$$\int_{-1}^{1} \left\{ \left(\frac{dv}{d\xi} \right)^2 - \lambda(Q - 1 - \varphi)e^{\varphi} v^2 \right\} d\xi \ge q \int_{-1}^{1} v^2\, d\xi \tag{9}$$

holds.

It is also not difficult to see that any bounded solution of problem (5), (6), defined for all $t \ge 0$ and constructed from $w_0(\xi)$, where $|w_0(\xi)| < \varepsilon$, and ε is a sufficiently small number (depending on the maximum of ρ and on the number q), satisfies the estimate $|w(\xi, t)| < \varepsilon(1 + \varepsilon)$. Now, for sufficiently small ε, it is easy to show that

$$\frac{\partial}{\partial t} \int_{-1}^{1} w^2(\xi, t)\, d\xi \le -\frac{q}{2} \int_{-1}^{1} w^2(\xi, t)\, d\xi \quad \text{or} \quad \int_{-1}^{1} w^2(\xi, t)\, d\xi \le ce^{-qt/2},$$

where c is a constant.

Thus, it has been proved that for $z(0) < 0$ (or $df/dy_0 < 0$) the stationary solution is stable. From the fact that, for prescribed values of the parameters Q, λ, there cannot exist more than three stationary solutions, and also

From the fact that $df/dy_0|_{y_0=0} > 0$, $df/dy_0 \to +\infty$ as $y_0 \to Q$, it follows that, in the case of the existence of three stationary solutions, two of them are stable. Let these be u_1, u_2. Then the solution $u_\varepsilon(\xi, t)$ of problem (3), (4) such that $u_\varepsilon(\xi, 0) = \varepsilon u_1 + (1-\varepsilon)u_2$, by what has been proved, as $t \to \infty$ and for every ε $(0 < \varepsilon < 1)$, converges to a stationary solution. It is easy to see that the sets

$$T_i \left\{ \varepsilon,\ u_\varepsilon(\xi, t_k) \xrightarrow[t_k \to \infty]{L_2} \hat{u}_i(\xi) \right\}$$

are open and do not intersect; therefore there exists ε_0 such that $u_{\varepsilon_0}(\xi, t) \overset{c}{\to} u_3(\xi)$, where $u_3(\xi)$ is the third stationary solution, and in any neighborhood of ε_0 there is an ε_1 such that $u_{\varepsilon_1}(\xi, t) - u_3(\xi)$ does not converge to zero. In the case of the existence of two stationary solutions (this is possible if λ is a bifurcation point of the corresponding boundary-value problem for the nonlinear ordinary differential equation), by similar reasoning we are convinced that one of them is unstable. It is obvious that if there exists only one solution, then it is stable. Thus the question of stability is resolved for all admissible values of the parameters.

We note that the necessity of the condition $z(0) \leq 0$ for stability can be proved directly, using the fact that for $z(0) > 0$, as follows from the comparison theorem, there exists a positive eigenvalue of the first problem for equation (7), i.e., that in the first approximation such a stationary solution is unstable.

It is not difficult to observe that the study of the general case (1), (2) creates no difficulties of principle; the stationary solutions are found in the same way as in ([4],[5]), and the stability investigation is more cumbersome, but in the case when all Π_i are equal, it can be carried out by essentially the same method.

Using the results of work ([6]) and oscillation theorems, one can establish a stability criterion, analogous to that obtained, for a stationary solution of the equation $u_t = u_{xx} + \lambda f(u, u_x)$ under the conditions $\alpha u_x + \beta u|_{x=-1} = \gamma u_x + \delta u|_{x=1} = 0$, if one takes into account that a stationary solution whose derivative vanishes twice is unstable.

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Note: Figure translations are in progress. See original paper for figures.