ON RINGS OF OPERATORS IN A SPACE WITH INDEFINITE METRIC

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Abstract

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ON RINGS OF OPERATORS IN A SPACE WITH INDEFINITE METRIC

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A well-known theorem of L. S. Pontryagin asserts that a self-adjoint operator in a space of type Π_n $(0 < n < \infty)$ has an n-dimensional nonnegative invariant subspace ([1]). This result was extended by M. A. Naimark to a family of mutually commuting normal operators ([2]). In the present article a generalization of these results is obtained for symmetric representations in a space of type Π_n of rings with involution having a complete system of representations whose dimensions do not exceed the number $m < \infty$. Another (very simple) proof of L. S. Pontryagin' s theorem is also obtained.

Theorem 1. Let T_x be a symmetric representation in a space H of type Π_n of a ring R with involution, having a complete system of representations whose dimensions do not exceed $m < \infty$. Then in H there exists a subspace H_1, invariant with respect to all T_x $(x \in R)$, such that $\dim H_1 \leq mn$, and the subspace $H_2 = \{x : (x, y) = 0 \text{ for all } y \in H_1\}$ is nonnegative.

Theorem 2. An irreducible ring of operators in a space H of type Π_n, closed in the strong topology and containing, together with each operator, also its adjoint (in the indefinite metric), coincides with the ring of all bounded operators in H.

We pass to the proof.

Lemma 1*. Let C be a family of bounded operators in a space H of type Π_n, and suppose that from $A \in C$ it follows that $A^* \in C$, and that C has no null invariant subspaces in H. Then

$$H = H^0 \oplus H^1 \oplus \cdots \oplus H^m, \tag{1}$$

where H^0 is positive, H^i $(1 \leq i \leq m)$ is either a space of type Π_{k_i} or a k_i-dimensional negative subspace, with $\sum k_i = n$; moreover $AH_j \subseteq H_j$ $(A \in C,\ 0 \leq j \leq m)$, and the restriction of C to H^i $(1 \leq i \leq m)$ is irreducible. The summand H^0 in (1) may be absent.

Proof. Note that, under the conditions of the lemma, the family C cannot have degenerate invariant subspaces. (Indeed, if H_0 is a degenerate invariant subspace, then $H_{0,0} = \{x : x \in H_0,\ (x, y) = 0 \text{ for all } y \in H_0\}$ will be an invariant null subspace.)

Suppose that there exist positive invariant subspaces in H. The set L of such subspaces is partially ordered by inclusion. If $\{H_\gamma\}$ is a chain in L, then the subspace $\mathcal{H} = \bigcup H_\alpha$ is invariant with respect to C and nonnegative and, by the preceding remark, nondegenerate; hence $\mathcal{H} \in L$, so that the chain $\{H_\gamma\}$ has an upper bound $\mathcal{H}$. By Zorn' s lemma, there exists in H a maximal positive invariant subspace H^0.

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* This lemma was obtained by the author in the article ([6]); at the same time an analogous assertion was obtained by M. A. Naimark.

In the subspace $H_1 = H \oplus H^0$ (invariant with respect to C) there are neither positive nor degenerate subspaces. Hence it follows that C in H_1 is either irreducible, or there is a decomposition $H_1 = H_{1,1} \oplus H_{1,2}$, where $H_{1,i}$ $(i = 1, 2)$ is invariant and is either a space of type Π_{s_i} or an s_i-dimensional negative subspace, with $0 < s_i < n$, $s_1 + s_2 = n$. If the restriction of C to $H_{1,i}$ $(i = 1, 2)$ is reducible, then analogous reasoning leads to a decomposition $H_{1,i} = H_{1,i,1} \oplus H_{1,i,2}$.

Continuing this reasoning, we arrive (in no more than n steps) at the required decomposition.

Lemma 2. *A self-adjoint operator A in a space H of type Π_n $(0 < n < \infty)$ has a nontrivial invariant subspace.*

Proof. If (x, y) is the indefinite scalar product in H, then

$$(x, y) = [(E - 2P)x, y],$$

where $[x, y]$ is the usual scalar product in H, and P is a projection operator, with $\dim PH = n$.

If $A^* = A$ in the Π_n-metric and A^+ is the operator adjoint to A with respect to the scalar product $[x, y]$, then, obviously,

$$A^+ - A = 2A^*P - 2PA,$$

whence it follows that the operator $\operatorname{Im} A = i/2(A^+ - A)$ is finite-dimensional; it is known ([3]) that such an operator has a nontrivial invariant subspace.

The following lemma is equivalent to L. S. Pontryagin' s theorem.

Lemma 3. *A self-adjoint operator A in a space H of type Π_n has a nonpositive eigenvector.*

Proof. If A has a zero invariant subspace $\mathcal{H}_0$, then $\dim \mathcal{H}_0 \leq n$, and the assertion of the lemma is obvious. If there are no such subspaces in H, then H admits the decomposition indicated in Lemma 1, and the restriction of A to

H_i $(1 \leq i \leq m)$ has no nontrivial invariant subspaces. But then, by Lemma 2, $\dim H_i = 1$, as was required.

Lemma 4. *If C is an irreducible set of operators in a space H of type Π_n, and from $A \in C$ it follows that $A^* \in C$, and the operator B commutes with all $A \in C$, then $B = \lambda E$.*

Proof. We may assume that $B = B^*$ (otherwise we consider the operators $B + B^*$, $i(B - B^*)$). If λ is an eigenvalue of B (see Lemma 3) and $H_\lambda = \{x : Bx = \lambda x\}$, then, by the conditions of the lemma, $AH_\lambda \subseteq H_\lambda$ $(A \in C)$. By the irreducibility of C, $H_\lambda = H$, i.e. $B = \lambda E$, as was required.

Proof of Theorem 2. Let C be an irreducible ring of operators in H, and suppose that from $A \in C$ it follows that $A^* \in C$, B is a bounded operator in H, $\{f_j^0,\ 1 \leq j \leq N\}$ are elements of H, and $\varepsilon > 0$. We shall show that there exists an $A \in C$ such that

$$\|Af_i^0 - Bf_i^0\| < \varepsilon \qquad (1 \leq i \leq N).$$

Obviously, we may assume that the vectors f_j^0 are linearly independent.

Let $\mathcal{H} = H \oplus \cdots \oplus H$ (N summands). In a natural way $\mathcal{H}$ becomes a Π_{nN}-space. Further, let P_i be the operator acting by the formula

$$P_i(f_1 \dots f_N) = f_i.$$

The operators

$$B'(f_1 \dots f_N) = (Bf_1, \dots, Bf_N)$$

form in $\mathcal{H}$ a ring C' with a natural involution.

Let $\mathcal{H}_0$ be the closure in $\mathcal{H}$ of the set of vectors of the form

$$(Bf_1^0, \dots, Bf_N^0) \qquad (B \in C). \tag{2}$$

Obviously, $B'\mathcal{H}_0 \subseteq \mathcal{H}_0$ $(B' \in C')$, and $\mathcal{H}_0$ is a nondegenerate subspace (otherwise the subspace $\mathcal{H}_0^0 = \{x :\ x \in \mathcal{H}_0,\ (x, y) = 0 \text{ for all } y \in \mathcal{H}_0\}$ would be invariant with respect to $B' \in C'$ and $0 < \dim \mathcal{H}_0^0 \leq nN$; but then the subspace $P_i\mathcal{H}_0^0 \subset H$ would be invariant with respect to C and finite-dimensional for all $1 \leq i \leq N$, whence $P_i\mathcal{H}_0^0 = 0$, i.e. $\mathcal{H}_0^0 = 0$).

Define in $\mathcal{H}$ the operator Q as follows:

$$Qx = \begin{cases} x, & x \in \mathcal{H}_0, \\ 0, & x \in \mathcal{H} \ominus \mathcal{H}_0. \end{cases}$$

Then $QA = AQ$ $(A \in C)$. Let $\{Q_{ij},\ 1 \leq i, j \leq N\}$ be the matrix corresponding to Q. Clearly, $Q_{ij}A = AQ_{ij}$ $(A \in C)$, whence, by Lemma 4, $Q_{ij} = \lambda_{ij}E$. At the same time

$$Q(f_1^0 \dots f_N^0) = (f_1^0 \dots f_N^0),$$

i.e. $\sum \lambda_{ij} f_j = f_i$ $(1 \le i \le N)$, whence, by the linear independence of $\{f_j\}$, $\lambda_{ij} = 0$ for $i \ne j$, $\lambda_{ii} = 1$, i.e. $Q = E$. This means that the system of vectors of the form (2) is dense in $\mathcal{H}$, which proves the theorem.

Proof of Theorem 1. We shall use the following result of Godement (4): if a ring R has a complete system of representations of dimension $\le m$, then every completely irreducible representation of R has dimension $\le m$. Complete irreducibility means that every bounded operator in the representation space is a strong limit of operators T_x $(x \in R)$. However, Godement' s proof remains valid if the condition of complete irreducibility is replaced a priori by the weaker one: the strong closure of the ring $\{T_x\}$ coincides with the ring of all operators in the representation space.

Returning to Theorem 1, suppose first that there are no null invariant subspaces in H. Then the decomposition indicated in Lemma 1 holds, and, by Godement' s theorem and Theorem 2, $\dim H_i \le m$, which proves the theorem.

If there are null invariant subspaces in H, and H_0 is some maximal one among them, then $\dim H_0 = k \le N$ and the subspace

$$H_0^{\perp}\{x \in H,\ x \perp H_0\}$$

is invariant with respect to T_x $(x \in R)$, with $H_0 \subset H_0^{\perp}$. The quotient space

$$\widehat{H}_0 = H_0^{\perp}/H_0$$

is naturally transformed into a space of type Π_{n-k} (5), and in $\widehat{H}_0$ there arises a representation $\widehat{T}_x$ that no longer contains null invariant subspaces. But then in $\widehat{H}_0$ there exists an invariant subspace of dimension $\le m(n-k)$. Thus in $H_0^{\perp}$ there exists an invariant subspace of dimension

$$k + (n-k)m \le nm,$$

satisfying the requirements of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.