ON ONE GENERALIZATION OF PERFECT MAPPINGS

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Abstract

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ON ONE GENERALIZATION OF PERFECT MAPPINGS

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We shall say that a space X (Hausdorff and completely regular) or a mapping f (continuous) has property P at the n-th infinity (or simply at infinity, when $n = 1$) if the Čech growth of order n of the space X or of the mapping f has property P (the definition of the growth of order n of a space or mapping is given in ([3])). We shall denote the growth of order n of a mapping f by $c^n(f)$, where $c^0(f) = f$, and the Čech growth of order n by $\gamma^n(f)$*.

A mapping f of a space X into a space Y will be called a K_n-mapping if the inverse image $f^{-1}(y)$ of each point $y \in Y$ is a space of the class K_n (the definition of the class K_n is given in ([3])). A compact mapping is a K_0-mapping, and a K_1-mapping may naturally be called a locally compact mapping.

A mapping f of a space X onto a space Y will be called n-perfect if the Čech growths $\gamma^0(f), \gamma^1(f), \ldots, \gamma^n(f)$ of the mapping f are closed mappings and f is a K_n-mapping. It is clear that a 0-perfect mapping is a perfect mapping (see ([1]), ([2])), while a closed mapping perfect at infinity (see ([3])) is a 1-perfect mapping.

Using the results of ([3]), one can prove the following theorem.

Theorem 1. *A mapping of a space from K_n onto an arbitrary space is a K_n-mapping and is perfect at the n-th infinity.*

Because of lack of space, the proofs of this and of the remaining theorems are not given.

Theorem 2. *A 1-perfect mapping X onto Y is a closed mapping, perfect at infinity.*

Theorem 2 is proved with the aid of the following lemma.

Lemma 1. *If f is a closed mapping of X into Y, and $\bar{f} : bX \to bY$ is an extension of f to compact extensions bX and bY, then the equality*

$$\bar{f}^{-1}(y) \cap (bX \setminus X) = \bar{f}^{-1}(y) \setminus f^{-1}(y)$$

holds for every $y \in Y$.

From this lemma, in particular, Theorem 4 of ([3]) also follows.

With the aid of Lemma 1 one can prove the following theorem.

Theorem 3. *A closed mapping f of a space X onto a space Y is a K_n-mapping if and only if f is a K_{n-1}-mapping at infinity.*

Using Theorems 2 and 3, we obtain the following theorem.

Theorem 4. *Let f be a mapping of X onto Y such that the mappings $\gamma^0(f), \gamma^1(f), \dots, \gamma^n(f)$ are closed. Then f is an n-perfect mapping if and only if f is perfect at the n-th infinity.*

In particular, for $n = 1$ we obtain Theorem 7 of ([3]).

* "A space or mapping has property P at the zeroth infinity" means that the space or mapping has property P.

With the aid of Lemma 1 one can prove the following theorem.

Theorem 5. Let f be a mapping of X onto Y such that the mappings $\gamma^0(f), \dots, \gamma^n(f)$ are closed. Then the mappings $c^0(f), \dots, c^n(f)$ are closed, and $R[\gamma^i(f)]$ is a closed subset of $R[\gamma^{i-1}(f)]$, where $1 \leq i \leq n$*.

The following theorems follow from Theorem 5.

Theorem 6. Let f be a mapping of X onto Y such that the mappings $\gamma^0(f), \dots, \gamma^{n-1}(f)$ are closed. Then, if some extension of order n of the mapping f is perfect, all extensions of order n are perfect.

Theorem 7. Let f be a mapping of X onto Y such that the mappings $\gamma^0(f), \dots, \gamma^{n-1}(f)$ are closed. Then f is an n-perfect mapping if and only if, for every point $x \in R^{(n+1)/2}(X)$, the set $f^{-1}[f(x)] \cap R^{(n-1)/2}(X)$ is compact, if n is odd, and the mapping induced by the mapping f on $R^{n/2}(X)$ is perfect, if n is even.

In particular, for $n = 1$ we obtain Theorem 6 from ([3]).

Let f be a mapping of X onto Y and g a mapping of Y onto Z. It is easy to see that, if f is a perfect mapping, then the mapping $fg : X \to Z$ is a K_n-mapping if and only if g is a K_n-mapping.

Using Theorems 5 and 8 from ([3]), one can prove the following theorems.

Theorem 8. Let f be a closed mapping such that $R(f)$ is a space in K_n. Then, if fg is a K_1-mapping, g is a K_{n+2}-mapping; and if fg is a K_2-mapping, then for even n, g is a K_{n+2}-mapping, while for odd n, g is a K_{n+3}-mapping.

Theorem 9. Let f be a closed, closed at infinity, locally compact mapping, and let $R(f)$ be compact. Then: 1) if g is a K_n-mapping, then for even n, fg is a K_{n+1}-mapping, while for odd n, fg is a K_n-mapping; 2) if fg is a K_n-mapping, then for even n, g is a K_n-mapping, while for odd n, g is a K_{n+1}-mapping.

With the aid of Theorems 4 and 7 we obtain the following theorem.

Theorem 10. Let f be an n-perfect mapping of X onto Y, where n is even. Then, if Y is paracompact, finally compact, or a space in K_m, then X is, in the n-th infinity, respectively paracompact, finally compact, or a space in K_m.

By virtue of one result from ([3]), it follows that if some extension of order n of a space has the perfect property, then all extensions of order n of this space have the same property. Using this fact and Theorem 4 from ([3]), one can prove the following theorem.

Theorem 11. If f is a closed mapping of a locally compact space X onto a space Y, and $R(f)$ is compact, paracompact, or finally compact in the n-th infinity, then Y is, in the $(n+2)$-th infinity, respectively compact, paracompact, or finally compact.

With the aid of Lemma 17 of E. G. Sklyarenko from ([4]) (see also Theorem 3.6 from ([5])) we obtain the following theorem.

Theorem 12. The space X is finally compact in the n-th infinity if and only if $R^{n/2}(X)$ is finally compact, if n is even, and $R^{(n-1)/2}(X)$ is a space of type $\mathfrak{S}$, if n is odd**.

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* If f is a mapping of X into Y, then $R(f)$ is the set of all points $y \in Y$ such that $f^{-1}(y)$ is noncompact (see ([3])).

** The definition of a space of type $\mathfrak{S}$ is given in ([4]).

We note that a finally compact or paracompact space is, in any even infinity, finally compact or paracompact, respectively.

Theorem 13. Let f be a closed mapping of X onto Y, and suppose that each point of $R(f)$ has a compact neighborhood. Then, if X is locally compact or is a space of type $\mathfrak{S}$, then Y is locally compact or a space of type $\mathfrak{S}$, respectively.

Theorem 14. Let f be a closed mapping of X onto Y, perfect at infinity, and suppose that $R(f)$ is finally compact. Then, if Y is a space of type $\mathfrak{S}$, then X is a space of type $\mathfrak{S}$.

Theorem 15. Let f be a closed mapping of X onto Y, closed at infinity, and suppose that each point of $R(f)$ has a compact neighborhood. Then, if X is paracompact at infinity, then Y is also paracompact at infinity.

We give one more theorem, generalizing Theorem 1 of E. G. Sklyarenko from ([6]) on the extension of perfect mappings to compact extensions.

Theorem 16. Let bX and bY be compact extensions of the spaces X and Y, with bX perfect and bY having a punctiform growth. Every closed mapping f of the space X into Y such that $R(f)$ is compact,

$$\dim R(f) = 0$$

and each point of $R(f)$ has a compact neighborhood in $\overline{f(X)}$, extends to a mapping $\bar{f} : bX \to bY$.

This theorem is proved with the aid of Lemma 1.

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Note: Figure translations are in progress. See original paper for figures.