ON THE APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY ALGEBRAIC POLYNOMIALS ON AN INTERVAL

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Abstract

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ON THE APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY ALGEBRAIC POLYNOMIALS ON AN INTERVAL

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1. Let $E_n(f)$ be the best uniform approximation of a function $f(x)$, continuous on the interval $[-1,1]$, by algebraic polynomials of degree not exceeding n. If $\mathfrak{M}$ is some class of functions, then we set

$$E_n[\mathfrak{M}] = \sup_{f\in\mathfrak{M}} E_n(f) \qquad (n = 0,1,2,\ldots).$$

In the case of approximation of 2π-periodic functions by trigonometric polynomials of order $\leq n$ on the whole axis, we shall write correspondingly $E_n^*(f)$, $E_n^*[\mathfrak{M}]$.

We denote by $W^{(r)}H_\omega$ ($r = 0,1,2,\ldots,\ W^{(0)}H_\omega = H_\omega$) the class of functions $f(x)$, defined on the interval $[-1,1]$, for which the modulus of continuity $\omega(f^{(r)};t)$ of the r-th derivative does not exceed a prescribed modulus of continuity $\omega(t)$. If $\omega(t) = Kt^\alpha$ $(0 < \alpha \leq 1)$, we shall write $W^{(r)}KH^{(\alpha)}$. The corresponding classes of 2π-periodic functions will be denoted by $W^{(r)}H_\omega^*$, $W^{(r)}KH_*^{(\alpha)}$.

Exact estimates of the quantities $E_n^*[W^{(r)}H_\omega^*]$, obtained in papers ([1,2]) for the case $\omega(t) = Kt$, $r = 0,1,2\ldots$, and in papers ([3–6]) for an arbitrary concave upward modulus of continuity $\omega(t)$ and $r = 0,1,2,3$, can be written in the form

$$E_{n-1}^*[W^{(r)}H_\omega^*] = E_{n-1}^*(f_{nr}) = \max_x |f_{nr}(x)| \qquad (n = 1,2,\ldots), \tag{1}$$

where $f_{nr}(x) = f_{nr}(\omega;x)$ is a function of period $2\pi/n$ with mean value over a period equal to zero, whose derivative of order r ($r = 0,1,2,\ldots$) is defined by the equalities

$$f_{nr}^{(r)}(x) = \begin{cases} \frac{1}{2}\,\omega(2x), & (0 \le x \le \pi/2n), \\ -\frac{1}{2}\,\omega(-2x), & (-\pi/2n \le x \le 0), \end{cases}$$

$$f_{nr}^{(r)}(x+\pi/n) = -f_{nr}^{(r)}(x).$$

Theorem 1. *For any concave upward modulus of continuity $\omega(t)$ and for each $r = 0, 1, 2, \ldots$, there exists a sequence of functions $g_n(x) \in W^{(r)}H_\omega$ such that*

$$E_{n-1}(g_n) \ge (1-\varepsilon_n)\max|f_{nr}(x)|,$$

where $\varepsilon_n \ge 0$, $\varepsilon_n = O(1/\ln n)$.

For the class $KH^{(1)}$ this assertion was proved earlier by S. M. Nikol' skii ([7]). By means of Theorem 1 one can show that, for arbitrary concave $\omega(t)$ and $r = 0, 1, 2, 3$, the upper bound $E_{n-1}[W^{(r)}H_\omega]$ is asymptotically equal to the right-hand side of equality (1). The fact of asymptotic coincidence of the quantities $E_n[W^{(r)}KH^{(\alpha)}]$ and $E_n^*[W^{(r)}KH_*^{(\alpha)}]$ $(r = 0, 1, 2, \ldots)$ was established in papers ([7–9]).

2. S. M. Nikol' skii, in 1946, was the first to discover one important feature of approximation of functions on an interval. He showed ([7]) that for any function $f \in KH^{(1)}$ one can construct a sequence of algebraic polynomials $P_n(f;x)$ of degree $\le n$ (depending linearly on f) such that, for all $x \in [-1, 1]$,

$$\begin{aligned} |f(x) - P_{n-1}(f;x)| &\le \frac{K\pi}{2n}\sqrt{1-x^2} + |x|\,O(\ln n/n^2) \\ &= E_{n-1}[KH^{(1)}]\{\sqrt{1-x^2} + o(1)\}, \end{aligned}$$

and this relation holds uniformly with respect to x. Thus the polynomials $P_n(f;x)$, while giving asymptotically best approximation on the class $KH^{(1)}$, give a significantly smaller deviation at the ends of the interval $[-1, 1]$. Later this result was generalized ([10]) to the classes $W^{(r)}KH^{(1)}$ $(r = 1, 2, \ldots)$.

It turns out that an analogous fact also holds for other, broader, classes of functions.

Theorem 2. *For any function $f \in KH^{(\alpha)}$ $(0 < \alpha < 1)$ there exists a sequence of algebraic polynomials $P_n(f;x)$ of degree $\le n$ such that, uniformly in $x \in [-1, 1]$, the inequality*

$$\begin{aligned} |f(x) - P_{n-1}(f;x)| &\le \frac{K}{2}\left(\frac{\pi}{n}\sqrt{1-x^2}\right)^{\alpha} + O(n^{-3/2\alpha}) \\ &= E_{n-1}[KH^{(\alpha)}]\{(1-x^2)^{\alpha/2} + o(1)\}. \end{aligned} \tag{2}$$

Theorem 3. *Whatever convex modulus of continuity $\omega(t)$ may be, for any function $f \in H_\omega$ there exists a sequence of algebraic polynomials $P_n(f;x)$ of degree $\leq n$ such that, uniformly in $x \in [-1,1]$,*

$$|f(x) - P_{n-1}(f;x)| \leq \frac{1}{2}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right) + o\left(\omega\left(\frac{1}{n}\right)\right). \tag{3}$$

The constant $1/2$ on the right-hand side cannot be decreased.

Theorem 4. *Whatever convex modulus of continuity $\omega(t)$ may be, for any function $f \in W^{(1)}H_\omega$ there exists a sequence of algebraic polynomials $P_n(f;x)$ of degree $\leq n$ such that, uniformly in $x \in [-1,1]$,*

$$|f(x) - P_{n-1}(f;x)| \leq \frac{1}{4}\int_0^{\frac{\pi}{n}\sqrt{1-x^2}} \omega(t)\,dt + o\left(\frac{1}{n}\omega\left(\frac{1}{n}\right)\right). \tag{4}$$

The constant $1/4$ on the right-hand side cannot be decreased.

Let us note that the polynomials $P_n(f;x)$ referred to in Theorems 2–4 depend on the function $f(x)$ nonlinearly.

The proof of Theorems 2–4 is based on the idea of intermediate approximation of functions of the classes H_ω (respectively $W^{(1)}H_\omega$) by continuous functions having almost everywhere a first (respectively, second) derivative bounded by a certain majorant depending on x. In this way asymptotically exact estimates are also obtained. For example, the following assertion holds.

Theorem 5. *If $\omega(t)$ is an arbitrary convex modulus of continuity, then for any function $f \in H_\omega$ one can specify a sequence of polygonal functions $\varphi_n(x)$ $(n = 1, 2, ...)$ such that:*

1) *for almost all $x \in [-1,1]$,*

$$|\varphi_n'(x)| \leq M_n(x) = \frac{1}{2}\left\{\omega_+'\left(\frac{\pi}{n}\sqrt{1-x^2}\right) + \omega_-'\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\right\} \quad (n = 1, 2, ...);$$

* The right-hand sides of inequalities (3) and (4) asymptotically coincide respectively with $E_{n-1}[H_\omega]$ and $E_{n-1}[W^{(1)}H_\omega]$ for $x = 0$.

2) at each point x of the interval $[-1,1]$, uniformly in x,

$$|f(x) - \varphi_n(x)| \leqslant \frac{1}{2}\max_{0\leqslant t\leqslant 2}[\omega(t) - tM_n(x)] + o\left(\omega\left(\frac{1}{n}\right)\right).$$

3. In paper (11) it is proved that, for any function $f(x)$ continuous on the interval $[-1,1]$, the inequalities

$$E_{n-1}(f) \leqslant \omega(f; \pi/n) \qquad (n = 1, 2, \ldots). \tag{5}$$

hold. From analogous considerations and Theorem 3 it follows that there exists a sequence of polynomials $P_n(f; x)$ such that

$$|f(x) - P_{n-1}(f; x)| \leqslant \omega\left(f; \frac{\pi}{n}\sqrt{1 - x^2}\right) + o\left(\omega\left(f; \frac{1}{n}\right)\right) \qquad (-1 \leqslant x \leqslant 1). \tag{6}$$

Estimates (5) and (6) are, as the following theorem shows, in a certain sense unimprovable.

Theorem 6. Whatever the number $\delta > 0$, there exists a sequence $f_n(x)$ of functions continuous on $[-1, 1]$ for which

$$E_{n-1}(f_n) \geqslant (1 - \varepsilon_n)\,\omega\left(f_n; \frac{\pi - \delta}{n}\right),$$

where $\varepsilon_n \geqslant 0$, $\varepsilon_n = O(1/\ln n)$.

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Note: Figure translations are in progress. See original paper for figures.