ON THE THEORY OF EXTREMAL PROBLEMS FOR QUASICONFORMAL MAPPINGS OF CLOSED RIEMANN SURFACES

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.54

MATHEMATICS

S. L. KRUSHKAL

ON THE THEORY OF EXTREMAL PROBLEMS FOR QUASICONFORMAL MAPPINGS OF CLOSED RIEMANN SURFACES

(Presented by Academician M. A. Lavrent' ev, 12 II 1966)

In this note a method is given which makes it possible to obtain the solution of a broad class of extremal problems for quasiconformal mappings. The following general problem is considered, relating the Teichmüller problem ([2]) to an extremal problem studied by P. P. Belinskii ([1]).

Problem A. In a given homotopy class α of quasiconformal homeomorphisms $w = f(z)$ of a closed Riemann surface S of genus $g > 1$ onto a surface S', with maximal dilatation $K[f] \leq q$, find the maximum of the real functional

$$F[f] = F(w_1, \dots, w_n), \qquad w_j = f(z_j) = u_j + iv_j,$$

where F is a function on the surface S'*.

The indicated class of mappings is denoted by $Q(\alpha, q)$ and is assumed to be nonempty.

Let the surfaces S and S' be represented in the form $S = U/\Gamma$, $S' = U'/\Gamma'$, where U and U' are the disks $|z| < 1$ and $|w| < 1$, and Γ and Γ' are Fuchsian groups of the first kind fixed for the corresponding surfaces S and S'. Then F becomes an automorphic function in the disk U'. We shall assume that the function F is continuously differentiable with respect to u_j, v_j, and that

$$\sum_{j=1}^{n} |F_{w_j}| > 0.$$

By $B(\Gamma')$ we denote the Banach space of Beltrami differentials $\mu(w)d\bar{w}/dw$ on S' with norm

$$\|\mu\|_{B(\Gamma')} = \text{vrai}\max_{w \in U'} |\mu(w)|.$$

Theorem 1. *There exists an analytic quadratic differential $\psi_0\, dw^2$ on S' and a constant γ, $0 \leq \gamma \leq 2\pi$, such that the quasiconformal mapping $w = f_0(z)$ of the* *disk U onto U' with characteristic of the inverse mapping $z = f_0^{-1}(w)$, satisfying the relations*

$$|\mu(w)| = (q-1)/(q+1), \qquad \arg\mu(w) = -\arg[e^{i\gamma}\psi^*(w) + \psi_0(w)], \tag{1}$$

where

$$\psi^*(w) = \sum_{j=1}^{n} F_{w_j} \sum_{A\in\Gamma'} \frac{{A'}^2(w)}{w_j - Aw}, \tag{2}$$

is an extremal mapping for which

$$\max F(w_1, \dots, w_n)$$

is attained in the class $Q(\alpha, q)$.

Proof. By virtue of the compactness of the class $Q(\alpha, q)$ there exists an extremal mapping $f_0(z) \in Q(\alpha, q)$ for which

$$F[f_0] = \max_{f\in Q(\alpha,q)} F(w_1, \dots, w_n).$$

Let $f_0(z_j) = w_{0j}$. By the Teichmüller theorem ([2]-[4]), there exists, and moreover a unique, quasiconformal mapping

$$w = f_1(z) \in Q(\alpha, q)$$

with the least maximal dilatation $K[f_1] = K_1$ among all homeomorphisms $f : S \to S'$ from the given homotopy class α that carry the points z_j into w_{0j}. In this case the characteristic

* It is assumed that the homeomorphisms f preserve orientation.

of the inverse mapping $z = f_1^{-1}(w)$ is equal to

$$\mu_1(w) = k_1\psi_1(w)/|\psi_1(w)|, \qquad k_1 = (K_1 - 1)/(K_1 + 1), \tag{3}$$

where $\psi_1 dw^2$ is a meromorphic quadratic differential on S', having at the points w_{0j} at most simple poles and determined up to a positive constant factor. It is clear that $F[f_1] = F[f_0]$.

We shall prove that $K_1 = q$. Let P' be a fixed fundamental polygon of the group Γ'. Denote by $L_1(U', \Gamma')$ the Banach space of all measurable quadratic differentials $\psi\, dw^2$ on S' with finite norm

$$\|\psi\|_{L_1(U',\Gamma')} = \iint_{S'} |\psi(w)|\, du\, dv, \qquad w = u + iv,$$

and by $A_1(U'/\Gamma')$ the subspace of $L_1(U', \Gamma')$ formed by analytic quadratic differentials on S'. Since the differential $\psi^*\, dw^2 \notin A_1(U', \Gamma')$, the distance from ψ^*

to $A_1(U', \Gamma')$ is equal to $d > 0$. There exists a linear functional v_0 in $L_1(U', \Gamma')$ such that

$$v_0(\psi) = 0 \quad (\psi \in A_1(U', \Gamma')); \qquad \|v_0\| = 1, \quad v(\psi^*) = d. \tag{4}$$

Here

$$v_0(\psi) = \iint_{P'} v_0(w)\psi(w)\, du\, dv,$$

where $v_0(w)$ is a function measurable in the disk U', such that

$$v_0(Aw) = v_0(w)A'(w)/\overline{A'(w)}, \qquad A \in \Gamma',$$

and

$$\operatorname*{vrai\,max}_{w \in U'} |v_0(w)| = 1.$$

The functional v_0 determines a mapping $\omega = H(w, \varepsilon)$ of the surface S' onto itself (a variation of the surface S') with characteristic $\varepsilon v_0(w) + O(\varepsilon^2)$, representable by the formula

$$\omega = H(\omega, \varepsilon) = w + \frac{\varepsilon}{\pi} \iint_{P'} v_0(\zeta) \sum_{A \in \Gamma'} \frac{{A'}^2(\zeta)}{w - A\zeta}\, d\sigma(\zeta) + O(\varepsilon^2), \tag{5}$$

where ε is a sufficiently small real parameter (see $(5, 6)$).

Denoting by $\mu^*(\omega)$ the characteristic of the mapping $z = f_1^{-1} \circ H^{-1}(\omega)$, we shall have

$$\tilde{\mu}(w) \equiv \mu^*(\omega(w))\overline{\omega_w}/\omega_w = \mu_1(w) - \varepsilon v_0(w) + \varepsilon\overline{v_0(w)}\mu_1^2(w) + O(\varepsilon^2) \tag{6}$$

and, consequently, in the case $K[f_1] < q$ the variation (5) is admissible for sufficiently small ε. On the other hand, expressing dw_j from (5), for $\varepsilon > 0$ we obtain

$$dF = \sum_{j=1}^{n} \left[F_{w_j}\, dw_j + \overline{F}_{w_j}\, d\overline{w}_j\right] = (2\varepsilon/\pi)\operatorname{Re} v_0(\psi^*) = 2\varepsilon d/\pi > 0, \tag{7}$$

which contradicts the extremality of the mapping $f_1(z)$. Therefore one must have $K_1 = q$, and then, by virtue of the uniqueness of the extremal Teichmüller mapping, all quasiconformal homeomorphisms $f : S \to S'$ from $Q(\alpha, q)$ taking the points z_j to w_{0j} coincide with $f_1(z)$.

Denote by Ω the subspace of $L_1(U', \Gamma')$ formed by elements of the form $\psi' = \lambda\psi^* + \psi$, where $\psi \in A_1(U', \Gamma')$, $\lambda = \text{const}$, and let

$$\sup_{\psi\in\Omega,\ \|\psi\|=1}\left|\iint_{P'}\mu_1(w)\psi(w)\,du\,dv\right| = k_1'. \tag{8}$$

We shall prove that $k_1' = k_1$, i.e. $\psi_1 \in \Omega$. Suppose that $k_1' < k_1$, and choose a number t so that $0 < t < k_1 - k_1'$. Then the norm of the functional

$$\mu_t \equiv \mu_1 - tv_0 = \iint_{P'}[\mu_1(w) - tv_0(w)]\psi(w)\,du\,dv \tag{9}$$

on the subspace Ω satisfies the inequality $\|\mu_t\|_\Omega \le k_1' + t = k_t < k_1$.

By the Hahn-Banach theorem, in $L_1(U',\Gamma')$ there is a linear functional $m_t(\psi)$ such that $m_t(\psi) = \mu_t(\psi)$ for $\psi \in \Omega$ and $\|m_t\|_{L_1(U',\Gamma')} = k_t$. Here

$$m_t(\psi) = \iint_{P'} m_t(w)\psi(w)\,du\,dv, \tag{10}$$

where $m_t(w)$ is measurable in U', is such that $m_t(Aw) = m_t(w)A'(w)/\overline{A'(w)}$, $A \in \Gamma'$, and $\operatorname{vrai\,max}_{w\in v'}|m_t(w)| = k_t$. Then the Beltrami differential

$$v_t(w) = \mu_1(w) - m_t(w)$$

defines in $L_1(U',\Gamma')$ a linear functional v_t satisfying the conditions

$$v_t(\psi) = 0, \quad \psi \in A_1(U',\Gamma'); \qquad v_t(\psi^*) = td > 0. \tag{11}$$

Consequently, $v_t(w)$ defines the corresponding variation $\omega = H(w,\varepsilon)$ of the surface S' with characteristic $\varepsilon v_t(w) + O(\varepsilon^2)$. The mapping $H(w,\varepsilon)$ is expressed by formula (5) with $v_0(w)$ replaced by $v_t(w)$. Denoting the characteristic of the mapping $z = f_1^{-1}\circ H^{-1}(\omega)$ by $\tilde\mu(\omega)$, we obtain

$$\tilde\mu(w) \equiv \tilde\mu^*(\omega(w))\,\omega_w/\omega_{\overline{w}} = \mu_1(w) - \varepsilon v_t(w) + \varepsilon\overline{v_t(w)}\mu_1^2(w) + O(\varepsilon^2). \tag{6'}$$

Applying (8)-(10) and (6′), we obtain that for $\varepsilon > 0$ the inequality

$$\sup_{\|\psi\|_{L_1(U',\Gamma')}=1}\left|\iint_{P'}\tilde\mu(w)\psi(w)\,du\,dv\right| < k_1 \tag{12}$$

holds, i.e. the variation with characteristic $\varepsilon v_t(w)$ is admissible. On the other hand, from (5) and (11) we shall have

$$dF = \sum_{j=1}^{n} \left[F_{w_j} \, dw_j + \overline{F}_{w_j} \, d\overline{w}_j \right] = (\varepsilon/\pi) \, 2 v_t(\psi^*) = 2\varepsilon t d/\pi > 0, \tag{13}$$

which contradicts the extremality of the mapping $w = f_1(z)$. Thus it is proved that $\psi_1 \, dw^2 \in \Omega$, whence the second relation in (1) follows.

The theorem is completely proved.

This method is also applicable to the solution of the Teichmüller problem [2–4] on finding $\min K[f]$ in a given homotopy class of homeomorphisms $f : S \to S'$, and gives a new solution of this problem.*

Let us indicate possible generalizations of Problem A. One may consider not only closed Riemann surfaces, but also compact surfaces bounded by a finite number of analytic curves, and require that the quasiconformal homeomorphisms under consideration take prescribed values at a finite number of prescribed points. Similarly to the Teichmüller problem (see (3)), this more general case is reduced to the case of closed surfaces studied above by constructing the double and introducing suitable two-sheeted and four-sheeted covering surfaces.

In particular, if Problem A is solved in the class of quasiconformal mappings $w = f(z)$ of the disk $|z| \le 1$ onto the disk $|w| \le 1$ with maximal dilatation $K[f] \le q$, carrying prescribed points $z^1, \dots, z^m$, $m \ge 2$, to prescribed points $w^1, \dots, w^m$, then the following holds.**

Theorem 2. There exist constants γ, $0 \le \gamma \le 2\pi$, and a function $\psi_0(w)$, analytic in the disk $|w| \le 1$ for $w \ne w^i$, $i = 1, \dots, n$, and having, in general, simple poles at the points $w^1, \dots, w^m$, such that the quasiconformal mapping $w = f_0(z)$ of the disk $|z| \le 1$ onto the disk $|w| \le 1$, with characteristic of the inverse mapping $z = f_0^{-1}(w)$, satisfies the relations:

$$|\mu(w)| = (q-1)/(q+1), \qquad \arg \mu(w) = -\arg \left[e^{i\gamma} \psi^*(w) + \psi_0(w) \right], \tag{14}$$

* This solution was found in a work of the author which will be published elsewhere.

** For $m = 2$, at least one of the points z^1, z^2 must lie inside the disk $|z| < 1$.

where

$$\psi^*(w) = \sum_{j=1}^{n} \frac{F_{w_j}}{w_j - w}, \tag{15}$$

is the extremal mapping for which the maximum of the real functional $F(w_1, \dots, w_n)$, $w_j = f(z_j)$, is attained in the class of quasiconformal mappings $w = f(z)$ of the disk $|z| \le 1$ onto the disk $|w| \le 1$, with maximal dilatation

$K[f] \leq q$, taking prescribed values $w^1, \dots, w^m$ at prescribed points $z^1, \dots, z^m$. In addition, the quadratic differential $e^{i\gamma}\psi^*(d)\,dw^2 + \psi_0(w)\,d\overline{w}^2$ must take real values on the circle $|w| = 1$.

If the number of fixed points z^i is equal to three when passing to the double of the disk $|z| \leq 1$, then the set of admissible variations is enlarged, and in this case, as shown in (1), the function $\psi_0(w)$ and the constant γ can be expressed through $w_1, \dots, w_n$.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
4 II 1966

CITED LITERATURE

1. P. P. Belinskii, *Sibirsk. matem. zhurn.*, **1**, No. 3, 303 (1960).
2. O. Teichmüller, *Abh. Preuss. Akad. Wiss., Math.-Naturwiss. Kl.*, **22**, 1 (1940).
3. L. Ahlfors, Collected Works.
4. L. Bers, *Spaces of Riemann surfaces and quasiconformal mappings*, IL, 1961, p. 104.
5. L. V. Ahlfors, L. Bers, *Ann. Math.*, **72**, No. 2, 385 (1960).
6. L. V. Ahlfors, *Ann. Math.*, **74**, No. 1, 171 (1961).

Note: Figure translations are in progress. See original paper for figures.