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ON SUFFICIENT CONDITIONS FOR OPTIMALITY

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Abstract

Full Text

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*CYBERNETICS
AND CONTROL THEORY*

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ON SUFFICIENT CONDITIONS FOR OPTIMALITY

(Presented by Academician L. S. Pontryagin on 8 V 1965)

This note considers the problem of automatic control in the case when the control region depends on the phase coordinates. The theorems obtained are directly adjacent to the work of V. G. Boltyanskii on the sufficiency of the maximum principle and were obtained under his direction.

In an n -dimensional space X , let an open set A and a piecewise-smooth set $N \subset A$ of dimension $\leq n - 1$ be considered. Denote by A_1, A_2, \dots, A_s such open sets in X , possibly disconnected, that $\bigcup A_i = A \setminus N$, and for any i_1, i_2 the sets A_{i_1} and A_{i_2} either coincide or do not intersect.

Let O be a certain domain of an r -dimensional space U , and for each $i = 1, 2, \dots, s$ let a function $G_i(x, u)$ be given, assumed to be defined, continuous, and continuously differentiable with respect to all arguments in some neighborhood of the set $A_i \times O$.

Further, we require that for any $x \in \bar{A}_{i_1} \cap \bar{A}_{i_2} \cap \dots \cap \bar{A}_{i_\alpha}$ the set $W(x) \subset O$ of those points $u \in O$ for which $G_{i_1}(x, u) \leq 0, G_{i_2}(x, u) \leq 0, \dots, G_{i_\alpha}(x, u) \leq 0$ be nonempty, and that if for some $u \in W(x)$ (in the appropriate numbering)

$$G_{i_1}(x, u) = G_{i_2}(x, u) = \dots = G_{i_\beta}(x, u) = 0,$$

$$G_{i_{\beta+1}}(x, u) < 0, \dots, G_{i_\alpha}(x, u) < 0,$$

then all the vectors

$$\text{grad}_u G_{i_1} = \{\partial G_{i_1}(x, u)/\partial u^1, \partial G_{i_1}(x, u)/\partial u^2, \dots, \partial G_{i_1}(x, u)/\partial u^r\},$$

.....

$$\text{grad}_u G_{i_\beta} = \{\partial G_{i_\beta}(x, u)/\partial u^1, \partial G_{i_\beta}(x, u)/\partial u^2, \dots, \partial G_{i_\beta}(x, u)/\partial u^r\}$$

are distinct from zero and lie (if they are regarded as emanating from a single point) inside some convex cone $K(x, u) \subset U$ that contains no entire straight line.

We shall consider phase trajectories of the controlled object

$$\dot{x}^i = f^i(x, u), \quad i = 1, 2, \dots, n,$$

or, in vector form,

$$\dot{x} = f(x, u).$$

(see (3)). The functions $f^1(x, u), f^2(x, u), \dots, f^n(x, u)$ are assumed to be defined, continuous, and continuously differentiable with respect to all arguments for $(x, u) \in A \times O$.

A piecewise-continuous, piecewise-smooth vector function $u(t) = (u^1(t), u^2(t), \dots, u^r(t))$, $t_0 \leq t \leq t_1$, will be called a control admissible with respect to the point $x_0 \in A$ if the phase trajectory of the equation $\dot{x} = f(x, u(t))$ with initial condition $x(t_0) = x_0$ belongs to A and $u(t) \in W(x(t))$ for $t_0 \leq t \leq t_1$. We shall say of such a control that it transfers the phase point x_0 to the position $x_1 = x(t_1)$. An admissible control with respect to the point x_0 that transfers this point to the po-

sition x_1 optimal if the time $t_1 - t_0$ of motion (under the action of this control) from the point x_0 to the point x_1 does not exceed the time of motion from x_0 to x_1 under the action of any other control admissible with respect to the point x_0 .

Lemma 1. Let $x_0 \in A$, $u(t)$, $t_0 \leq t \leq t_1$, be a control admissible with respect to the point x_0 , and let $\varepsilon > 0$. Then there exists a point $\bar{x}_0 \in A$ and a control $\bar{u}(t)$, $t_0 \leq t \leq t_1$, admissible with respect to \bar{x}_0 , such that for every $t \in [t_0, t_1]$ one has $|x(t) - \bar{x}(t)| < \varepsilon$, $\bar{u}(t) \in \text{Int } W(\bar{x}(t))$.

Theorem 1. Let $u(t)$, $t_0 \leq t \leq t_1$, be a control admissible with respect to the point $x_0 \in A$, carrying the phase point x_0 into the position x_1 ; let $x(t)$ be the corresponding trajectory. For the optimality of the control $u(t)$ it is sufficient that there exist a piecewise-smooth set $M \subset A$ of dimension $\leq n - 1$ and a continuous on A function $\omega(x) = \omega(x^1, x^2, \dots, x^n)$, which on the set $A \setminus (M \cup N)$ is continuously differentiable with respect to x^1, x^2, \dots, x^n and satisfies the conditions

$$\sum_{\alpha=1}^n \frac{\partial \omega(x)}{\partial x^\alpha} f^\alpha(x, u) \leq 1$$

for $x \in A \setminus (M \cup N)$, $u \in W(x)$, $\omega(x_1) - \omega(x_0) = t_1 - t_0$.

Theorem 1, as well as the analogous Theorem 1 from (2), is proved by methods of the theory of smooth manifolds (4).

In the article (1) the notion of regular synthesis was introduced. In the case under consideration (in connection with the fact that a variable domain of control is being considered), separate points of the definition given there had to be somewhat modified.

Let piecewise-smooth sets $R \subset A$ of dimension $\leq n - 1$, piecewise-smooth sets $P^0 \subset P^1 \subset \dots \subset P^{n-1} \subset P^n$, for which $P^n = A$, $P^{n-1} \supset N$, and a vector-function $v(x)$, defined in A so that $v(x) \in W(x) \subset O$, be given. We shall say that these sets and the function $v(x)$ realize a **regular synthesis** for the equation $\dot{x} = f(x, u)$ in the domain A , if the following conditions are fulfilled:

A. $|f(x, v(x))| \neq 0$ for $x \in A$.

B. Each component of the set $P^i \setminus (P^{i-1} \cup R)$, $i = 1, 2, \dots, n$, is an i -dimensional smooth manifold in A ; these components will be called i -dimensional cells. The function $v(x)$ is continuous and continuously differentiable on each cell and can be extended to a continuously differentiable function in some neighborhood of the cell. All cells are divided into cells of the first and second kind. All n -dimensional cells are of the first kind. Cells of the first kind also include all $(n - 1)$ -dimensional cells that entirely belong to N . All zero-dimensional cells, except the point a , are cells of the second kind.

C. If σ is some i -dimensional cell of the first kind ($i \geq 1$), then through each point of this cell there passes a unique trajectory of the equation

$$\dot{x} = f(x, v(x))$$

(the distinguished trajectory), passing along the cell σ . There exists an $(i - 1)$ -dimensional cell $\Pi(\sigma)$ of the first or second kind such that every trajectory of this system going in the cell σ reaches $\Pi(\sigma)$ in finite time; moreover, different trajectories from the cell σ approach different points of the cell $\Pi(\sigma)$. If the cell $\Pi(\sigma)$ is of the second kind, then the trajectories going along the cell σ meet the cell $\Pi(\sigma)$, striking it at a nonzero angle. If, however, $\Pi(\sigma)$ is a cell of the first kind, then such trajectories may be tangent to the cell $\Pi(\sigma)$. The points at which trajectories meet the first-kind cell $\Pi(\sigma)$ while being tangent to it will be called **points of degenerate switchings**.

D. If σ is some i -dimensional cell of the second kind ($i \geq 0$), then there exists an $(i + 1)$ -dimensional cell of the first kind such that from any point

from the cell σ there emanates a unique trajectory of the system $\dot{x} = f(x, v(x))$, passing along the cell $\Sigma(\sigma)$, and the function $v(x)$ is continuous and continuously differentiable on $\sigma \cup \Sigma(\sigma)$.

D. This item remains the same as in paper (1), if the domain V is denoted by A and the set N by R .

E. Denote by $T(x_0)$ the time of motion along the indicated trajectory with velocity $f(x, v(x))$ from the point $x_0 \in A$ to the point a . Then $T(x)$ is a continuous function of $x \in A$.

We shall call a regular synthesis **optimal** if, whatever the point $x \in A$ and whatever control $u(t)$ admissible with respect to this point, transferring the phase point from the position x to a in time $T_u(x)$, one always has

$$T(x) \leq T_u(x).$$

The set of points from which there emanate indicated trajectories containing points of degenerate switchings will be denoted by Q .

Lemma 2. *The function $T(x)$ is differentiable with respect to x^1, x^2, \dots, x^n at all points of the set $A \setminus (P^{n-1} \cup R \cup Q)$.*

Suppose that a regular synthesis has been carried out in the domain A . Put $\omega(x) = -T(x)$. By Lemma 2, the function $\omega(x)$ is continuously differentiable in the domain $A \setminus (P^{n-1} \cup R \cup Q)$.

Lemma 3. *If the function $\omega(x)$ satisfies the relation*

$$\sup_{u \in W(x)} \sum_{\alpha=1}^n \frac{\partial \omega(x)}{\partial x^\alpha} f^\alpha(x, u) \leq 1 \quad \text{for } x \in A \setminus (P^{n-1} \cup R \cup Q),$$

then the constructed synthesis is optimal.

Suppose that a regular synthesis is constructed in the domain A and $x_0 \in A$. Denote by

$$l = \{x(t), t_0 \leq t \leq t_1\}$$

the indicated trajectory emanating from the point x_0 . The segment $[t_0, t_1]$ is divided by certain points

$$\tau_0 = t_0 < \tau_1 < \dots < \tau_q = t_1$$

into a finite number of pieces in such a way that, for any $\mu = 1, 2, \dots, q$, the arc

$$l_\mu = \{x(t), \tau_{\mu-1} < t < \tau_\mu\}$$

of the trajectory l passes entirely along a cell of the first kind, which we denote by σ_μ . Along each arc l_μ there is defined a function $v(x)$, continuous and continuously differentiable in σ_μ . For any point $x \in \sigma_\mu$, by the definition of a regular synthesis,

$$G_j(x, v(x)) \leq 0 \quad (j = 1, 2, \dots, s).$$

Moreover, if $\sigma_\mu \subset N$, then for all $x \in \sigma_\mu$ and $j = 1, 2, \dots, s$ one has

$$G_j(x, v(x)) < 0;$$

whereas if $\sigma_\mu \subset N$, then for all $x \in \sigma_\mu$ and for certain

$$1 \leq j_1 < j_2 < \dots < j_\alpha \leq s$$

one has

$$G_{j_1}(x, v(x)) = G_{j_2}(x, v(x)) = \dots = G_{j_\alpha}(x, v(x)) = 0.$$

In the latter case we denote by $P(\sigma_\mu)$ the cell of greatest dimension containing the cell σ_μ and lying entirely in N .

Let a nonzero continuous vector-function

$$\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))$$

on each of the intervals $(\tau_{\mu-1}, \tau_\mu)$, $\mu = 1, 2, \dots, q$, satisfy the system of equations

$$\dot{\psi}_i = - \sum_{\alpha=1}^n \frac{\partial f^\alpha(x(t), v(x(t)))}{\partial x^i} \psi_\alpha, \quad i = 1, 2, \dots, n,$$

if $\sigma_\mu \subset N$, and the system of equations

$$\dot{\psi}_i = - \sum_{\alpha=1}^n \left(\frac{\partial f^\alpha(x(t), v(x(t)))}{\partial x^i} + \sum_{\beta=1}^r \frac{\partial f^\alpha(x(t), v(x(t)))}{\partial u^\beta} \left(\frac{\partial v^\beta(x(t))}{\partial x^i} \right)_{P(\sigma_\mu)} \right) \psi_\alpha,$$

$$i = 1, 2, \dots, n,$$

where $(\partial v^\beta / \partial x^i)_{P(\sigma_\mu)}$ denotes the derivative of v^β with respect to x^i in the manifold $P(\sigma_\mu)$, if $\sigma_\mu \subset N$. Put

$$H(\psi, x, u) = (\psi, f(x, u)) = \sum_{\alpha=1}^n \psi_\alpha f^\alpha(x, u).$$

If one can choose a vector-function $\psi(t)$ satisfying the conditions listed above such that, for any fixed $t \in [t_0, t_1]$,

$$H(\psi(t), x(t), v(x(t))) = \max_{u \in W(x(t))} H(\psi(t), x(t), u) = \lambda \geq 0,$$

then we shall say that along the indicated trajectory l the maximum principle is satisfied.

Theorem 2. *A regular synthesis along each of whose indicated trajectories the maximum principle is satisfied is optimal.*

I consider it my pleasant duty to express my gratitude to Prof. V. G. Boltyanskii.

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named after P. Stučka

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Note: Figure translations are in progress. See original paper for figures.

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