TOWARD THE SOLUTION OF VARIATIONAL PROBLEMS OF NONCLASSICAL TYPE

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 519.3

MATHEMATICS

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TOWARD THE SOLUTION OF VARIATIONAL PROBLEMS OF NONCLASSICAL TYPE

(Presented by Academician B. N. Petrov, 19 VI 1965)

Let, among the coordinates $x = (x^1, \dots, x^N)$ of a system and its controls $u = (u^1, \dots, u^M)$, satisfying on the interval $t_0 \le t \le t_1$ the system of equalities

$$\dot{x} = f(x, u), \qquad f = (f^1, \dots, f^N) \tag{1}$$

and the system of inequalities

$$|g^l(x, u)| \le 1, \qquad l = 1, \dots, L, \tag{2}$$

as well as certain conditions at the endpoints,

$$h(x_0, x_1) = 0, \qquad h = (h^1, \dots, h^R), \qquad R \le 2N - 1, \tag{3}$$

it is required to find those which give the functional

$$J = J(x_0, x_1) \tag{4}$$

an extremal value. The coordinates x^n, $n = 1, \dots, N$, will be sought in the class of continuous functions, and the controls u^m, $m = 1, \dots, M$, in the class of piecewise-continuous functions. The case of a nonautonomous system is reduced to the present one by adjoining the coordinate $x^{N+1} = t$ and the relation (1) $\dot{x}^{N+1} = 1$. Let all the requirements of the calculus of variations imposed on the functions entering into (1)–(4) be fulfilled, and let the equalities (1) be valid also on the boundary of the domain (2).

Replace the system of inequalities (2) by the system of equalities

$$\varphi = s(x, u, v) - g(x, u) = 0, \tag{5}$$

where φ, s, g, and v are L-vectors. The functions $s^l,\ l = 1, \dots, L$, are not greater than unity in modulus and are twice differentiable in the open domain of the arguments. We shall show that, as the transformation functions s^l, it is expedient to use a trigonometric function, for example the sine, $s = (\sin v^1, \dots, \sin v^L)$. Indeed, in the case (*) of constraints $|u^1| \le 1$ or $(u^1)^2 + (u^2)^2 \le 1$, equality (5) will have the form

$$\varphi^1 = \sin v^1 - u^1 = 0 \quad \text{or} \quad \varphi^1 = \sin v^1 - 2\big[(u^1)^2 + (u^2)^2\big] + 1 = 0.$$

In the case of constraints of the form $\underline{\omega}^1(x,u) \le \omega^1(x,u) \le \overline{\omega}^1(x,u)$, transformation (5) may be written as follows:

$$\varphi^1 = \sin v^1 - \frac{2\omega^1(x,u) - \overline{\omega}^1(x,u) - \underline{\omega}^1(x,u)}{\overline{\omega}^1(x,u) - \underline{\omega}^1(x,u)} = 0.$$

Thus, the nonclassical problem on the extremum of the functional (4) under conditions (1), (2), (3) has been reduced to the problem of the classical calculus of variations on the extremum of the functional (4) under conditions (1), (5), (3).

If one introduces the multipliers $\lambda = (\lambda^1, \dots, \lambda^N)$, $\mu = (\mu^1, \dots, \mu^L)$ and the function $H(x,u,v,\lambda,\mu) = f\lambda + \varphi\mu$, then the vanishing of the first variation of the functional J entails the Euler–Lagrange equations:

$$H_x = -\dot{\lambda}, \qquad H_u = 0, \qquad H_v = 0, \qquad H_\lambda = \dot{x}, \qquad H_\mu = 0, \tag{6}$$

where, for example, $H_x = (H_{x^1}, \dots, H_{x^N})$, $H_{x^1} = \partial H / \partial x^1$, etc., for all x, u, v, λ and μ, the endpoint conditions:

$$dJ + [\lambda\, dx - H\, dt]_0^1 = 0, \qquad h(x_0, x_1) = 0 \tag{7}$$

and the Weierstrass-Erdmann conditions on the continuity of λ and H at points of discontinuity of the derivatives $\dot{x}$

$$\lambda_- = \lambda_+, \qquad H_- = H_+. \tag{8}$$

The study of a strong variation leads to consideration of the Weierstrass function E, which is equal to the decrease of the function H under any deviations $\tilde{u}$ and $\tilde{v}$, admissible by the equalities (1), (5), (3), from the extremal values u and v: $E = H(x,u,v,\lambda,\mu) - H(x,\tilde{u},\tilde{v},\lambda,\mu)$. For a strong minimum (maximum) of the functional J it is necessary that $E \ge 0$ ($\le$), or

$$H(x,u,v,\lambda,\mu) \ge H(x,\tilde{u},\tilde{v},\lambda,\mu)\ (\le), \tag{9}$$

i.e., that the function H have a maximum with respect to the admissible controls u and v. The condition of this maximum of H for the case of weak variations can be obtained by expanding the function H in a Taylor series. Then, discarding small quantities higher than second order, we obtain the Legendre–Clebsch condition necessary for a weak minimum (maximum) of the functional J:

$$\sum_{m=1}^{M}\sum_{i=1}^{M} H_{u^m u^i}\delta u^m \delta u^i + 2\sum_{m=1}^{M}\sum_{l=1}^{L} H_{u^m v^l}\delta u^m \delta v^l + \sum_{l=1}^{L}\sum_{j=1}^{L} H_{v^l v^j}\delta v^l \delta v^j \le 0\ (\ge). \tag{10}$$

If, as the transformation functions, one chooses sines, i.e., if $s = (\sin v^1, \dots, \sin v^L)$, then $s_v = \|\cos v^l\|_L^L$ will be a diagonal matrix, and $s^l_{v^l} = -\sin v^l = -s^l$, $l = 1, \dots, L$. Then from (6) the equations $s_v \mu = 0$ mean $\mu^l \cos v^l = 0$, or

$$\mu^l \sqrt{1-(g^l)^2} = 0, \qquad l = 1, \dots, L. \tag{11}$$

Consequently, at points where $\mu^l \ne 0$, $g^l = \pm 1$, i.e., the extremal lies on the l-th boundary of the domain of variation (2). Points where $\mu^l = 0$ may correspond to parts of the extremal lying inside the domain of variation and to moments of transition of the extremal from the upper boundary $g^l = 1$ to the lower $g^l = -1$, or conversely. For systems (1), (2) linear with respect to the controls u, the conditions (10) mean

$$-\mu^l \sin v^l \le 0\ (\ge) \quad \text{or} \quad \mu^l g^l \ge 0, \qquad l = 1, \dots, L\ (\le), \tag{12}$$

whence it follows that, for $\mu^l \ne 0$, the sign of g^l must be the same as the sign of μ^l, i.e., for $\mu^l > 0$, $g^l = 1$, while for $\mu^l < 0$, $g^l = -1$. Thus the control must be chosen so that, when μ^l passes through zero with a subsequent change of sign, the extremal also changes the sign of the boundary. *Isolated zeros of the Lagrange multipliers μ^l, $l = 1, \dots, L$, are switching points of relay controls, while the vanishing of the multiplier μ^l on an interval may correspond to smooth control, i.e., to a continuous transition from one boundary to the other.*

The transformation by means of the sine function maps the restricted domain of variation of the control function onto a multi-sheeted unrestricted domain of variation of the transformed control function, thereby eliminating the problem of one-sided variation and allowing the use of the necessary conditions of the classical calculus of variations. In particular, the condition (12), simple and convenient for determining the sequence of switching of the control, contains the theorem that *the transition of the extremal from one boundary (2) to another is determined by the behavior of the corresponding Lagrange multiplier.* This feature of the trigonometric transformation is a substantial advantag···

···compared with the other transformations ([2–4]) and was not noted earlier ([5,6]).

Let us proceed to present examples of problems that can be investigated by the method of the trigonometric transformation of variables. We shall consider problems illustrating, in the works ([1,7]), the maximum principle and the method of dynamic programming. In Example 1 a time-optimal system with a constraint on the control is considered; in Example 2, a problem with movable endpoints. In Example 3 an isoperimetric problem with a constraint on the angle of inclination of the extremal is considered. This example demonstrates a method for optimizing nonlinear systems, as well as the case when the solution is nonlinear in character. In Example 4, taken from ([8]), a time-optimal system is considered with simultaneous constraints on its control and coordinate.

Example 1. Find u such that, for $\dot{x}^1 = x^2,\ \dot{x}^2 = u,\ |u| \leqslant 1$, the time of motion of the system $J = t_1,\ t_0 = 0$, from the point (x_0^1, x_0^2) to the origin is minimal. Let $\varphi = \sin v - u = 0,\ \dot{x}^3 = 1$; then $H = x^2\lambda^1 + u\lambda^2 + \lambda^3 + (\sin v - u)\mu$. The Euler–Lagrange equations (6) will be

$$\dot{\lambda}^1 = 0, \quad \dot{\lambda}^2 = -\lambda^1, \quad \dot{\lambda}^3 = 0, \quad \lambda^2 - \mu = 0, \quad \mu\cos v = 0, \quad \dot{x}^1 = x^2, \quad \dot{x}^2 = u,$$

$$\dot{x}^3 = 1, \quad u = \sin v.$$

Having determined the Lagrange multipliers

$$\lambda^1 = C_1, \quad \lambda^2 = C_2 - C_1 t, \quad \lambda^3 = C_3,$$

we note that the Weierstrass-Erdmann condition (8) requires the same value of the constants C_1, C_2, and C_3 on the whole extremal. Since $\mu = \lambda^2$, $\mu \neq 0$ everywhere except, possibly, at the point $t_* = C_2/C_1$. Then the fifth Euler-Lagrange equation, or the equality (11), shows that the extremal lies on the boundary $u = \pm 1$ everywhere except, perhaps, at the point t_*. Consequently, as in ([1]), the control $u(t)$ is a piecewise-constant function taking the values ± 1 and having no more than two intervals of constancy. The condition (12) $\mu u \geqslant 0$ determines that the point t_* is the switching point of the control u. We note that condition (8) on the continuity of the function H at the corner points of extremals is satisfied, since at the switching point λ^2 vanishes. The Weierstrass condition for a strong minimum (9) is also satisfied for the chosen control u, for

$$(C_2 - C_1 t)u \geqslant (C_2 - C_1 t)\tilde{u}$$

for all $\tilde{u}$, $|\tilde{u}| \leqslant 1$. Integration of the equations of motion with the found u determines the optimal trajectories in the phase plane x^1x^2 ([1]).

Example 2. Keeping the conditions of Example 1, let us consider the problem of the fastest transition from the point (x_0^1, x_0^2) to the axis x^2 of the phase plane, i.e., the problem with a movable right endpoint under the condition (3) of the form $x_1^1 = 0$. Then the condition (7)

$$[\lambda^1 dx^1 + \lambda^2 dx^2 + (1 + \lambda^3)dx^3 - cdt]_0^1 = 0$$

at the right endpoint will determine $\lambda_1^2 = 0$. Since $\lambda^2 = C_2 - C_1 t$ can vanish only at one point, λ^2, and also μ, preserve a constant sign for $t_0 \leqslant t < t_1$, and the motion of the phase point along an arc of a parabola of one family will be optimal ($u = 1$ for $x_0^1 < 0$, or $u = -1$ for $x_0^1 > 0$) without switching. The assumption that, for some initial point, the extremal phase trajectory may also be a parabola of the other family gives, for another initial point, an extremal with switching, which is impossible.

Example 3. Isoperimetric problem. Find a curve $x^1(t)$ of given length c, connecting two fixed points (t_0, x_0^1) and (t_1, x_1^1), such that the area of the region between the curve and the t-axis is minimal, while the tangent of the angle of inclination of the curve to the t-axis in absolute value does not exceed unity. Let x^1 denote the ordinate of the curve, x^2 its length, and x^3 the area to be minimized. Then

$$\dot{x}^1 = u, \quad x_0^1 = a, \quad x_1^1 = b, \quad \dot{x}^2 = \sqrt{1 + (u)^2}, \quad x_0^2 = 0, \quad x_1^2 = c, \quad \dot{x}^3 = x^1, \quad x_0^3 = 0, \quad x_1^3 = J,$$

$$\varphi = \sin v - u = 0.$$

$$H = u\lambda^1 + \sqrt{1 + (u)^2}\,\lambda^2 + x^1\lambda^3 + (\sin v - u)\mu,$$

where a, b, and c are positive.

The conditions (6) will be:

$$\dot{\lambda}^3 = -\lambda^1, \quad 0 = -\lambda^2, \quad 0 = -\lambda^3, \quad \lambda^1 + \lambda^2 u/\sqrt{1 + (u)^2} - \mu = 0, \quad \mu\cos v = 0,$$

$$\dot{x}^1 = u; \quad \dot{x}^2 = \sqrt{1 + (u)^2}, \quad \dot{x}^3 = x^1, \quad u = \sin v,$$

whence

$$\lambda^1 = C_1 + C_3 t, \quad \lambda^2 = -C_2, \quad \lambda^3 = -C_3, \quad \mu = C_1 + C_3 t - \frac{C_2 u}{\sqrt{1 + (u)^2}}.$$

From the transversality condition (7) it follows that $C_3 = 1$. Setting $C_1 + t = \tau$ and $C_2 = \rho$, we obtain $\lambda^1 = \tau$, $\lambda^2 = -\rho$, $\lambda^3 = -1$, $\mu = \tau - \rho u/\sqrt{1 + (u)^2}$. By condition (11), for $\mu \neq 0$, $u = \pm 1$; for $\mu = 0$, $u = \tau/\sqrt{(\rho)^2 - (\tau)^2}$. If $u = -1$, then $x^1 = -\tau + A^1$, $x^2 = \sqrt{2}\tau + A^2$, $x^3 = -(\tau)^2/2 + A^1\tau + A^3$. If $u = 1$, then $x^1 = \tau + B^1$, $x^2 = \sqrt{2}\tau + B^2$, $x^3 = (\tau)^2/2 + B^1\tau + B^3$. If $u = \tau/\sqrt{(\rho)^2 - (\tau)^2}$, then $x^1 = -[(\rho)^2 - (\tau)^2]^{1/2} + C^1$, $x^2 = \rho \arcsin \tau/\rho + C^2$, $x^3 = C^1\tau - \frac{1}{2}\tau\sqrt{(\rho)^2 - (\tau)^2} - \frac{1}{2}(\rho)^2 \arcsin \tau/\rho + C^3$. All A, B, and C are constants of integration. Consequently, the sought curve can consist of no more than three pieces: two line segments with angles of inclination to the τ-axis equal to $\pm 45^\circ$, and an arc of a circle of radius ρ with center at the point $(0, C^1)$.

Condition (8), $H_- = H_+$, at the points τ_* of change of control requires

$$-\tau_* u_- + \rho\sqrt{1 + (u_-)^2} = -\tau_* u_+ + \rho\sqrt{1 + (u_+)^2}.$$

Therefore, for $\rho \neq 0$, it must be that $u_- = u_+$, i.e., the segments must be tangent to the circular arc at the joining points. Then

$$\pm 1 = \frac{\tau_*}{\sqrt{(\rho)^2 - (\tau_*)^2}},$$

whence

$$\tau_*^1 = -\frac{\rho}{\sqrt{2}}, \qquad \tau_*^2 = \frac{\rho}{\sqrt{2}}.$$

Condition (10),

$$\frac{\rho(1-(u)^2)}{\sqrt{(1+(u)^2)^3}} + \mu u \geq 0,$$

will be satisfied if, for $\tau_0 \leq \tau \leq -\rho/\sqrt{2}$, $u = -1$, $\mu = \tau + \rho/\sqrt{2}$; for $-\rho/\sqrt{2} \leq \tau \leq \rho/\sqrt{2}$, $u = \tau/\sqrt{(\rho)^2 - (\tau)^2}$, $\mu = 0$; and for $\rho/\sqrt{2} \leq \tau \leq \tau_1$, $u = 1$, $\mu = \tau - \rho/\sqrt{2}$. Condition (9) on the first, second, and third intervals is written in the form

$$-\tau - \frac{\rho}{\sqrt{2}} \geq \tilde{u}\tau - \rho\sqrt{1+(\tilde{u})^2}, \qquad \sqrt{(\rho)^2-(\tau)^2} \leq \rho\sqrt{1+(\tilde{u})^2} - \tilde{u}\tau, \qquad \tau - \frac{\rho}{\sqrt{2}} \geq \tilde{u}\tau - \rho\sqrt{1+(\tilde{u})^2},$$

which is fulfilled for any $|\tilde{u}| \leq 1$.

For the end points $t_0 = -2\sqrt{2}$, $a = 4$, $t_1 = 2\sqrt{2}$, $b = 4$, with curve length $c = 4 + \pi$, we shall have

$$\rho = 2, \qquad t_*^1 = -\sqrt{2}, \qquad t_*^2 = \sqrt{2}, \qquad A^1 = 4 - 2\sqrt{2}, \qquad B^1 = 4 - 2\sqrt{2}, \qquad C^1 = 4,$$

i.e., on the interval $-2\sqrt{2} \leq t \leq -\sqrt{2}$, $u = -1$, $x^1 = 4 - 2\sqrt{2} - t$; on $-\sqrt{2} \leq t \leq \sqrt{2}$, $u = t/\sqrt{4-t^2}$, $x^1 = 4 - \sqrt{4-t^2}$; on $\sqrt{2} \leq t \leq 2\sqrt{2}$, $u = 1$, $x^1 = 4 - 2\sqrt{2} + t$. The latter coincides with the result due to M. Hestenes (7).

Example 4. Suppose the coordinate x^1 in Example 1 is subject to the additional constraint $|x^1| \leq 1$, i.e., the transformation $\varphi^1 = \sin v^1 - u = 0$ must be supplemented by the transformation $\varphi^2 = \sin v^2 - x^1 = 0$. Then

$$H = x^2\lambda^1 + u\lambda^2 + \lambda^3 + (\sin v^1 - u)\mu^1 + (\sin v^2 - x^1)\mu^2.$$

Equations (6) will be

$$\dot{\mu}^2 = -\lambda^1, \qquad \dot{\lambda}^1 = -\lambda^2, \qquad 0 = \dot{\lambda}^3, \qquad \lambda^2 - \mu^1 = 0, \qquad \mu^1 \cos v^1 = 0, \qquad \mu^2 \cos v^2 = 0,$$

$$\dot{x}^1 = x^2, \qquad \dot{x}^2 = u, \qquad \dot{x}^3 = 1; \qquad u = \sin v^1, \qquad x^1 = \sin v^2.$$

The condition $\mu^2 = 0$, corresponding, according to (11), to parts of the extremal lying inside the domain $|x^1| \leq 1$, leads to $\lambda^1 = C_1$, $\lambda^2 = C_2 - C_1 t$, $\lambda^3 = C_3$, $\mu^1 = C_2 - C_1 t$, i.e., gives the extremals of Example 1. The case $\mu^2 \neq 0$ and $\cos v^2 = 0$ satisfies the sixth Euler-Lagrange equation and corresponds to $\sin v^2 = \pm 1$, i.e., to the motion of the phase point along the boundary $x^1 = \pm 1$. Consequently, the extremals in the phase plane consist of pieces of parabolas corresponding to Example 1 and, possibly, pieces of the boundary of the domain of variation of the coordinates. By virtue of condition (8), the constants C_1, C_2, and C_3 are the same for all parts of the extremal. Therefore, from condition (12), $\mu^1 u \geq 0$, it follows that the control is switched at the point where the sign of μ^1 changes, at the instant

$$t_* = C_2/C_1,$$

i.e., at the point of intersection of the parabolas satisfying the boundary conditions on the phase plane. Condition (9), as in Example 1, is fulfilled.

Received
19 VI 1965

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Note: Figure translations are in progress. See original paper for figures.