ON GREEN’ S FUNCTIONS FOR MODEL HAMILTONIANS

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Abstract

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In a number of our papers ([1,2]) it was established that, for model Hamiltonians of the Bardeen type, under sufficiently general conditions the free energy of the system per unit volume can be computed with asymptotic accuracy by introducing the so-called approximating Hamiltonians, which are quadratic forms in the Fermi amplitudes. In paper ([3]) a stronger result was obtained, namely, it was shown that the Green' s functions for the given model Hamiltonian and for the corresponding approximating Hamiltonian are also close. This result, however, was obtained only for the case of zero temperature. The aim of the present article is to generalize it to the case of temperature greater than zero.

Let us note that the question of estimating the difference of two Green's functions or two correlation functions—for the exact and the approximating Hamiltonians —constitutes a considerably more difficult problem than the question of the corresponding estimate for the free energies. Therefore we shall simplify the formulation of the problem as much as possible and shall consider a Hamiltonian of Bardeen type with a factorizing kernel

$$\Gamma = T - 2VI \cdot I^+ g - r(I + I^+)V; \tag{1}$$

here

$$T = \frac{p^2}{2m} - \mu; \qquad I = \frac{1}{2V} \sum_{(f)} \lambda_f a_f^+ a_{-f}^+;$$

μ is the chemical potential; a_f, a_f^+ are Fermi amplitudes; $f = (p, \sigma)$; $-f = (-p, -\sigma)$; σ is a spin index taking two values $(-1, +1)$; p is the usual quasiclassical momentum spectrum, passing into a continuous one in the limiting process $V \to \infty$; g is a positive parameter characterizing the intensity of the interaction; r is a positive parameter removing the statistical degeneracy, which in the final formulae we shall let tend to zero.

On the functions λ_f we impose the generally accepted conditions

$$\lambda_{-f} = -\lambda_f \tag{2}$$

and, for simplicity, set:

$$\lambda_f = \begin{cases} \text{a continuous function} & \text{for } |p^2/2m - \mu| \leq \Delta, \\ 0 & \text{for } |p^2/2m - \mu| > \Delta; \end{cases}$$

more general cases, considered in ([2]), may also be investigated.

The corresponding approximating Hamiltonian will be

$$\Gamma^0 = T - 2VCg(I^+ + I) + 2VC^2 g - r(I + I^+)V. \tag{3}$$

Here the quantity C is determined from the condition of the absolute minimum of the expression

$$-\frac{\theta}{V} \ln \mathrm{Sp}\, e^{-\Gamma^0/\theta}, \tag{4}$$

in view of which we have $C = \langle I \rangle$, where, in general,

$$\langle \mathcal{H} \rangle_H = \frac{\mathrm{Sp}\, \mathcal{H} e^{-H/\theta}}{\mathrm{Sp}\, e^{-H/\theta}}.$$

Let us write the equations of motion for the Hamiltonian (1)

$$\begin{aligned} i\, da_f/dt &= T_{fa} f - \lambda_f a^+_{-f}(2I^+ g + r), \\ i\, da^+_f/dt &= -T_{fa} f^+ + \lambda_f (2Ig + r) a_{-f}. \end{aligned} \tag{5}$$

We note that the equations of motion with the approximating Hamiltonian (3) differ from the equations of motion (5) in that on the right-hand side of (5), instead of the operators I, I^+, there stands C. On the other hand, the operators a_f, a^+_f in both equations coincide at $t = 0$, since then we have for them the usual Heisenberg representation. It is therefore obvious that if we wish to establish the closeness of the correlation functions $\langle A(t)B(\tau)\rangle_\Gamma \sim \langle A(t)B(\tau)\rangle_{\Gamma^0}$, where $A, B = a_f, a_{-f}, a^+_f, a^+_{-f}$, then it will be desirable for us to establish the closeness of the operator I to its mean value $C = \langle I \rangle_{\Gamma^0}$.

Consider the difference

$$a = f_{\Gamma^0} - f_\Gamma$$

of the free energies per unit volume for the Hamiltonians Γ^0, Γ, and note that

$$\partial a/\partial r = \langle I + I^+\rangle - 2C; \qquad \partial a/\partial g = 2(\langle II^+\rangle_\Gamma - C^2). \tag{6}$$

Consequently,

$$\langle (I - C)(I^+ - C)\rangle_\Gamma = \langle II^+\rangle_\Gamma - C\langle (I + I^+)\rangle_\Gamma + C^2 = \frac{1}{2}\frac{\partial a}{\partial g} - \frac{\partial a}{\partial r}C.$$

On the other hand, it is clear that

$$|I^+ I - II^+| \leq K/V,$$

where $K = \text{const}$.

Thus,

$$|\langle (I^+ - C)(I - C)\rangle_\Gamma| \leq \left|\frac{1}{2}\frac{\partial a}{\partial g} - \frac{\partial a}{\partial r}C\right| + \frac{K}{V}.$$

As is seen, the smallness of the deviation $(I - C)$ will be established if we show the smallness of the derivatives $\partial a/\partial g, \partial a/\partial r$.

In our above-mentioned works it was shown that

$$|a| \leq \mathcal{E}(1/V) \to 0 \quad \text{as } V \to \infty. \tag{7}$$

We shall now strengthen this result by establishing the smallness of the derivatives (6). To this end, note that

$$\partial^2 f_\Gamma/\partial g^2 \leq 0, \qquad \partial^2 f_\Gamma/\partial r^2 \leq 0,$$

therefore

$$\partial^2 a/\partial g^2 \geq \partial^2 f_{\Gamma^0}/\partial g^2, \qquad \partial^2 a/\partial r^2 \geq \partial^2 f_{\Gamma^0}/\partial r^2.$$

But f_{Γ^0} is computed elementarily, by diagonalizing Γ^0—a quadratic form in Fermi operators—and it is not hard to show that

$$\left|\partial^2 f_{\Gamma^0}/\partial g^2\right| \leq D(\delta), \qquad \left|\partial^2 f_{\Gamma^0}/\partial r^2\right| \leq D(\delta) \quad \text{for } r \geq \delta > 0, \tag{8}$$

where $D(\delta)$ is bounded uniformly with respect to $V \to \infty$ for any fixed $\delta > 0$. Thus,

$$\partial^2 a/\partial g^2 \geq -D(\delta), \qquad \partial^2 a/\partial r^2 \geq -D(\delta), \qquad r \geq \delta > 0. \tag{9}$$

From (7) and (9) it is not hard to obtain

$$|\partial a/\partial r| \leq 2\sqrt{2\mathcal{E}(1/V)D(\delta)}, \quad \partial a/\partial g \leq 2\sqrt{2\mathcal{E}(1/V)D(\delta)}; \qquad r \geq \delta,$$

therefore

$$\begin{aligned} &|\langle (I-C)(I^+-C)\rangle_\Gamma| \leq \mathcal{E}_0(1/V,\delta), \\ &|\langle (I^+-C)(I-C)\rangle_\Gamma| \leq \mathcal{E}_0(1/V,\delta); \qquad r \geq \delta; \end{aligned} \tag{10}$$

here $\mathcal{E}_0(1/V,\delta) \to 0$ as $V \to \infty$ for any fixed value $\delta > 0$.

Let us pass to new Fermi amplitudes

$$\alpha_f = u_f a_f + v_f a^+_{-f}, \quad \text{where} \quad u_f = \frac{1}{\sqrt{2}}\sqrt{1+\frac{T_f}{\Omega_f}};$$

$$v_f = \frac{-\varepsilon(f)}{\sqrt{2}}\sqrt{1-\frac{T_f}{\Omega_f}}, \quad \Omega_f = \sqrt{(2Cg+r)^2\lambda_f^2 + T_f^2}.$$

Then the equations of motion (5) take the form

$$i\, d\alpha_f^+/dt + \Omega_f \alpha_f^+ = R_f, \quad i\, d\alpha_f/dt - \Omega_f\alpha_f = -R_f^+,$$

where

$$R_f = 2\lambda_f g\{u_f(I-C)a^+_{-f} + v_f a_f^+(I^+-C)\}.$$

Taking into account the inequalities (10), one can show that

$$\left|\langle \alpha_f^+(t)\alpha_f(\tau)\rangle_\Gamma - e^{i\Omega_f(t-\tau)}\langle\alpha_f^+(0)\alpha_f(0)\rangle_\Gamma\right| \leq \sqrt{\mathcal{E}_1(1/V,\delta)}\,|t-\tau|,$$

$$\left|\langle \alpha_f(\tau)\alpha_f^+(t)\rangle_\Gamma - e^{i\Omega_f(t-\tau)}\langle\alpha_f(0)\alpha_f^+(0)\rangle_\Gamma\right| \leq \sqrt{\mathcal{E}_1(1/V,\delta)}\,|t-\tau|. \tag{11}$$

Moreover,

$$\begin{aligned} &\left|\langle \alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma - e^{-i\Omega_f(t+\tau)}\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma\right| \leq \\ &\leq \sqrt{\mathcal{E}_1(1/V,\delta)}(|t|+|\tau|+\mathcal{E}_1(1/V,\delta)|t|\,|\tau|), \end{aligned} \tag{12}$$

where $\mathcal{E}_1(1/V,\delta) \to 0$ as $V \to \infty$ for any fixed $\delta > 0$.

On the other hand, all the correlation averages under consideration, in particular $\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma$, depend on t and τ only through the difference $(t-\tau)$. Therefore

$$\langle\alpha_f(t)\alpha_{-f}(t)\rangle_\Gamma = \langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma,$$

and from (12) we have

$$\left|\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma \left|1-e^{-2i\Omega_f t}\right|\right| \leq 2\sqrt{\mathcal{E}_1(1/V,\delta)}\,|t| + \mathcal{E}_1(1/V,\delta)|t|^2.$$

Taking here the arbitrary t equal to $\pi/2\Omega_f$, we find

$$\left|\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma\right| \leq \sqrt{\mathcal{E}_1(1/V,\delta)}\,\frac{\pi}{2\Omega_f} + \mathcal{E}_1(1/V,\delta)\,\frac{\pi^2}{8\Omega_f^2}.$$

Consequently, from (12) we obtain

$$\left|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma\right| \leq \sqrt{\mathcal{E}_1(1/V,\delta)}\,\big\{|t-\tau| + \pi/2\Omega_f +$$

$$+\sqrt{\mathcal{E}_1(1/V,\delta)}\,\pi^2/8\Omega_f^2\big\}.$$

But, obviously,

$$\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_{\Gamma^0} = 0,$$

and therefore we may also write

$$\left|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma - \langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_{\Gamma^0}\right| \leq$$

$$\leq \sqrt{\mathcal{E}_1(1/V,\delta)}\left\{|t-\tau| + \pi/2\Omega_f + \sqrt{\mathcal{E}_1(1/V,\delta)}\,\pi^2/8\Omega_f^2\right\}. \tag{13}$$

Here, under the averaging sign, the operators $a_f,\ a_f^+$ satisfy the equations of motion for the Hamiltonians Γ and Γ^0, respectively, appearing in the corresponding averages here.

Let us now turn to inequalities (11). Using the spectral representations (4–6),

$$\langle\alpha_f^+(t)\alpha_f(\tau)\rangle_\Gamma = \int_{-\infty}^{+\infty} I(\omega)e^{i\omega(t-\tau)}\,d\omega, \quad \langle\alpha_f(\tau)\alpha_f^+(t)\rangle_\Gamma = \int_{-\infty}^{+\infty} I(\omega)e^{i\omega(t-\tau)}e^{\omega/\theta}\,d\omega,$$

in which $I(\omega) \geqslant 0$, we can show that

$$\left|\langle\alpha_f^+(t)\alpha_f(\tau)\rangle_\Gamma - \langle\alpha_f^+(t)\alpha_f(\tau)\rangle_{\Gamma^0}\right| \leqslant \mathcal{E}_2(1/V,\delta)|t-\tau| + \mathcal{E}_3(1/V,\delta), \qquad (14)$$

$$\left|\langle\alpha_f(\tau)\alpha_f^+(t)\rangle_\Gamma - \langle\alpha_f(\tau)\alpha_f^+(t)\rangle_{\Gamma^0}\right| \leqslant \mathcal{E}_2(1/V,\delta)|t-\tau| + \mathcal{E}_3(1/V,\delta),$$

where $\mathcal{E}_2(1/V,\delta)$, $\mathcal{E}_3(1/V,\delta)$ tend to zero as $V\to\infty$ for any fixed positive value of δ.

Returning to the original Fermi amplitudes a_f, a_f^+, from (11)–(14) we see that, in general,

$$|\langle A(t)B(\tau)\rangle_\Gamma - \langle A(t)B(\tau)\rangle_{\Gamma^0}| \leqslant \eta(1/V,\delta)|t-\tau| + \eta'(1/V,\delta), \qquad (15)$$

where $A, B = a_f, a_f^+, a_{-f}, a_{-f}^+$ and $\eta(1/V,\delta)\to 0$, $\eta'(1/V,\delta)\to 0$, $V\to\infty$ for any fixed value of $\delta > 0$.

We emphasize that relation (15), like all the preceding analogous relations, is valid when $r \geqslant \delta$.

The expression $\langle A(t)B(\tau)\rangle_{\Gamma^0}$ is computed elementarily, and it can be shown that

$$\lim_{r\to 0}\left\{\lim_{V\to\infty}\langle A(t)B(\tau)\rangle_{\Gamma^0}\right\} = \lim_{V\to\infty}\langle A(t)B(\tau)\rangle_{H^0}, \qquad (16)$$

where $H^0 = \Gamma^0(r=0)$.

Thus, from relation (15) there follows the existence of the limit

$$\lim_{r\to 0}\left\{\lim_{V\to\infty}\langle A(t)B(\tau)\rangle_\Gamma\right\}, \qquad (17)$$

equal to (16).

But (17) is nothing other than the quasiaverage (in the sense of (3))

$$\{A(t)B(\tau)\}_H \qquad (18)$$

for the Hamiltonian $H = \Gamma(r=0)$.

Thus, we see that the quasiaverage (18) is asymptotically equal to the quasiaverage for the approximating Hamiltonian H^0. Hence analogous relations for the Green' s functions follow immediately.

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Note: Figure translations are in progress. See original paper for figures.