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Abstract

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MATHEMATICS

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ON THE ASYMPTOTIC PROPERTIES OF THE RESOLVENT OF AN INTEGRAL EQUATION WITH A KERNEL DEPENDING ONLY ON DIFFERENCES OF THE VARIABLES

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Let D be a bounded domain of the n -dimensional Euclidean space R , with boundary satisfying the Lyapunov conditions and strictly star-shaped with respect to the origin of coordinates. Denote by D_ρ the domain similar to D , with coefficient of similarity equal to ρ . Consider the integral equation

$$y(P) + \lambda \int_{D_\rho} K(P - Q)y(Q) d\sigma_Q = f(P) \quad (P \in D_\rho). \quad (1)$$

Here P and Q are vectors, and $d\sigma$ is the element of volume in R .

We shall speak of the asymptotic behavior of the resolvent of the integral equation (1) as $\rho \rightarrow +\infty$ and for fixed $\lambda > 0$, and partly as $\lambda \rightarrow +\infty$ and for fixed $\rho > 0$.

For the kernel $K(P)$ we shall henceforth assume the following conditions to be satisfied:

1. The Fourier transform of the function $K(P)$ (understood in the generalized sense, say, as a functional in the class of finite functions) is a nonnegative function $a(S)$, integrable in every closed part of the space.
2. The function $a(S)$ defined in 1 satisfies the condition

$$\int_R \frac{a^2(S)}{1 + a(S)} d\sigma_S < \infty.$$

By $\Gamma_\rho(P, Q; \lambda)$ we shall denote the resolvent of equation (1), and by $\Gamma(P - Q; \lambda)$ the resolvent of the limiting equation

$$y(P) + \lambda \int_R K(P - Q)y(Q) d\sigma_Q = f(P). \quad (1^*)$$

In addition, let us denote

$$\Gamma_\rho^{(2)}(P, Q; \lambda) = \frac{1}{\lambda} [K(P - Q) - \Gamma_\rho(P, Q; \lambda)];$$

$$\Gamma^{(2)}(P; \lambda) = \frac{1}{\lambda} [K(P) - \Gamma(P; \lambda)].$$

The functions $\Gamma(P; \lambda)$ and $\Gamma^{(2)}(P; \lambda)$, corresponding to the limiting equation, are easily found. As is known,

$$\Gamma(P; \lambda) = (2\pi)^{-n} \int_R \frac{a(S)}{1 + \lambda a(S)} e^{i(PS)} d\sigma_S; \quad \Gamma^{(2)}(P; \lambda) = (2\pi)^{-n} \int_R \frac{a^2(S)}{1 + \lambda a(S)} e^{i(PS)} d\sigma_S$$

(by condition 2, the first integral converges in the mean-square sense, while the second converges absolutely).

Theorem 1. For any $\rho > 0$ and $\lambda > 0$ we have the inequality

$$\Gamma_\rho^{(2)}(P, P; \lambda) \leq \Gamma^{(2)}(0; \lambda) = (2\pi)^{-n} \int_R \frac{\alpha^2(S)}{1 + \lambda \alpha(S)} d\sigma_S,$$

and, as $\rho \rightarrow +\infty$ and for fixed $\lambda > 0$, the asymptotic formula

$$\int_{D_\rho} \Gamma_\rho^{(2)}(P, P; \lambda) d\sigma_P \sim \frac{V(D_\rho)}{(2\pi)^n} \int_R \frac{\alpha^2(S)}{1 + \lambda \alpha(S)} d\sigma_S$$

holds.

($V(D_\rho)$ is the volume of the domain D_ρ .)

Remark. Generally speaking, the asymptotic formula of Theorem 1 cannot be extended to the case where $\lambda \rightarrow +\infty$ while $\rho > 0$ is fixed. However, for kernels $K(P)$ satisfying, for every $\mu > 0$, the condition

$$K(\mu P) = \mu^\beta K(P)$$

with some real constant β , the assertion of Theorem 1 is valid as $\lambda^2 + \rho^2 \rightarrow +\infty$ ($\lambda \geq \delta > 0$, $\rho \geq \delta > 0$).

Let us note in particular the simplest result of this special type, obtained for the case $K(P) = |P|^{-\gamma}$ after computing all integrals occurring in the formula (for the Fourier transform of the function $|P|^{-\gamma}$, see, for example, (1)).

Theorem 2. Let $\psi_\gamma(P, Q; \lambda)$ be the resolvent of the integral equation

$$y(P) + \lambda \int_D |P - Q|^{-\gamma} y(Q) d\sigma_Q = f(P) \quad \left(0 < \gamma < \frac{n}{2}\right).$$

Let, furthermore,

$$\psi_\gamma^{(2)}(P, Q; \lambda) = \frac{1}{\lambda} [|P - Q|^{-\gamma} - \psi_\gamma(P, Q; \lambda)];$$

and let $N_\gamma(t)$ be the number of eigenvalues of our equation lying on the interval $(-t, 0)$. Then the asymptotic formulas

$$\int_D \psi_\gamma^{(2)}(P, P; \lambda) d\sigma_P \sim \frac{\pi}{(n - \gamma) \sin \pi\gamma/(n - \gamma)} 2^{-\gamma} \pi^{-\pi/2} \frac{\Gamma((n - \gamma)/2)}{\Gamma(\gamma/2)} \Omega_n \lambda^{\gamma/(n - \gamma) - 1} V(D) \quad (\lambda \rightarrow +\infty),$$

$$N_\gamma(t) \sim \frac{1}{n} \cdot 2^{-\gamma} \pi^{-\pi/2} \frac{\Gamma((n - \gamma)/2)}{\Gamma(\gamma/2)} \Omega_n t^{n/(n - \gamma)} V(D) \quad (t \rightarrow +\infty)$$

are valid.

(Ω_n is the area of the unit sphere in R .)

We shall set forth the main idea of the proof of Theorem 1, without dwelling on questions of justifying the admissibility of one or another formal operation.

Denote by $\rho(P)$ the value of the parameter ρ for which the boundary of the domain D_ρ passes through the point P . Put

$$\Pi_X(S) = (2\pi)^{-n/2} e^{i(XS)} + (2\pi)^{-n/2} \int_{D_{\rho(X)}} a_X(T) e^{i(TS)} d\sigma_T, \quad (2)$$

where the functions $a_X(P)$ are solutions of the integral equation

$$a_X(P) + \lambda \int_{D_{\rho(X)}} K(P - Q) a_X(Q) d\sigma_Q = -\lambda K(P - X) \quad (P \in D_{\rho(X)}).$$

It is not difficult to verify that the functions $\{\Pi_X(S)\}$ form a system orthogonal in the whole space with weight $1 + \lambda\alpha(S)$, i.e., that

$$\int_R \Pi_X(S) \overline{\Pi_T(S)} [1 + \lambda\alpha(S)] d\sigma_S = \delta(X - T) \quad (X \in R, \quad T \in R).$$

Indeed, writing the orthogonality condition for the function $\Pi_X(S)$ and the exponential $e^{i(P S)}$ for $P \in D_{\rho(X)}$, we arrive at the integral written above—equation, and the normalization is guaranteed by the first term in the expression for the function $\Pi_X(S)$.

It is also not difficult to show that

$$e^{i(P S)} = \left\{ \int_{D_{\rho(P)}} b_P(X) \Pi_X(S) d\sigma_X \Pi_{\rho} + (S) \right\} (2\pi)^{n/2},$$

where

$$\begin{aligned} b_P(X) &= \int_R e^{i(P S)} \overline{\Pi_X(S)} [1 + \lambda\alpha(S)] d\sigma_S = \\ &= \lambda \int_{D_{\rho(X)}} K(P - Q) a_X(Q) d\sigma_Q + \lambda K(P - X) \quad (X \in D_{\rho(P)}). \end{aligned}$$

From this integral representation for the exponential it is easy to obtain that, for the function

$$\Phi(S) = \int_{D_{\rho}} e^{i(P S)} \varphi(P) d\sigma_P$$

the integral representation

$$\Phi(S) = \int_{D_{\rho}} b(X) \Pi_X(S) d\sigma_X, \quad (3)$$

is valid, where

$$b(X) = \int_R \Phi(S) \overline{\Pi_X(S)} [1 + \lambda\alpha(S)] d\sigma_S.$$

Now we shall express the solution of equation (1) in terms of the functions $\Pi_X(S)$. For this purpose denote

$$Y(S) = \int_{D_\rho} y(P) e^{i(PS)} d\sigma_P.$$

Recalling that

$$\alpha(S) = \int_R K(P) e^{i(PS)} d\sigma_P,$$

we can write equation (1) in the form

$$(2\pi)^{-n} \int_R Y(S) [1 + \lambda \alpha(S)] e^{-i(PS)} d\sigma_S = f(P) \quad (P \in D_\rho). \quad (4)$$

We shall find the coefficients of the expansion of the function $Y(S)$ in the functions of the orthogonal system constructed by us, and then restore $Y(S)$ by means of the integral representation (3). Denoting these coefficients by $b(X)$, we obtain, according to formulas (2) and (4),

$$b(X) = f(X) + \int_{D_\rho(X)} \overline{a_X(T)} f(T) d\sigma_T,$$

and, according to (3),

$$Y(S) = \int_{D_\rho} \left[f(X) + \int_{D_\rho(X)} \overline{a_X(T)} f(T) d\sigma_T \right] \Pi_X(S) d\sigma_X.$$

Substituting into the last formula, in place of the function $\Pi_X(S)$, its expression from formula (2) and applying the inverse Fourier transform, we find

$$\begin{aligned} y(P) &= f(P) + \int_{D_\rho(P)} \overline{a_P(T)} f(T) d\sigma_T + \int_{D_\rho - D_\rho(P)} a_X(P) f(X) d\sigma_X + \\ &+ \int_{D_\rho - D_\rho(P)} a_X(P) \int_{D_\rho(X)} \overline{a_X(T)} f(T) d\sigma_T d\sigma_X. \end{aligned}$$

From the formula found for the solution of equation (1) we easily obtain the following formulas for its resolvent:

$$\Gamma_\rho(P, Q; \lambda) = -\frac{1}{\lambda} a_P(\overline{Q}) - \frac{1}{\lambda} \int_{D_\rho - D_\rho(Q)} a_X(P) \overline{a_X(Q)} d\sigma_X \quad (P \in D_\rho(Q));$$

$$\Gamma_\rho(P, Q; \lambda) = -\frac{1}{\lambda} a_Q(P) - \frac{1}{\lambda} \int_{D_\rho - D_\rho(P)} \overline{a_X(P)} a_X(Q) d\sigma_X \quad (P \in D_\rho - D_\rho(Q)).$$

We shall show that the integral

$$\int_R a_X(P) \overline{a_X(Q)} d\sigma_X$$

converges absolutely. For this purpose, note that the functions $c(X) = a_X(P)$ are the coefficients in the expansion of the function

$$F_P(S) = \frac{\lambda a(S)}{1 + \lambda a(S)} e^{i(P S)}$$

with respect to the functions of our orthogonal system $\{\Pi_X(S)\}$. Therefore Bessel's inequality gives us

$$\int_R |a_X(P)|^2 d\sigma_X \leq \int_R |F_P(S)|^2 [1 + \lambda a(S)] d\sigma_S = \lambda^2 \int_R \frac{a^2(S)}{1 + \lambda a(S)} d\sigma_S,$$

whence, by Schwarz's inequality, we obtain

$$\int_R |a_X(P) \overline{a_X(Q)}|^2 d\sigma_X \leq \lambda^2 \int_R \frac{a^2(S)}{1 + \lambda a(S)} d\sigma_S.$$

Consequently, the function $\Gamma_\rho(P, Q; \lambda)$, as $\rho \rightarrow +\infty$, tends to a limit, which can only be the function $\Gamma(P - Q; \lambda)$, the resolvent of equation (1*). Moreover,

$$\Gamma_\rho^{(2)}(P, P; \lambda) = \Gamma^{(2)}(0; \lambda) - \frac{1}{\lambda^2} \int_{R - D_\rho} |a_X(P)|^2 d\sigma_X,$$

whence we obtain the inequality of Theorem 1.

To obtain the asymptotic formula, we estimate the last integral more precisely. By virtue of the extremal property of orthogonal systems, we have the inequality

$$\frac{1}{\lambda^2} \int_{R - D_\rho} |a_X(P)|^2 d\sigma_X \leq \inf \int_R \left| \frac{\alpha(S)}{1 + \lambda \alpha(S)} e^{i(P S)} - \Phi_\rho(S) \right|^2 [1 + \lambda \alpha(S)] d\sigma_S,$$

where the greatest lower bound is taken over all functions $\Phi_\rho(S)$ of the form

$$\Phi_\rho(S) = \int_{D_\rho} e^{i(QS)} \varphi(Q) d\sigma_Q.$$

Denote by $D_{\rho,P}$ the domain obtained from the domain D_ρ by translating the origin of coordinates to the point P , and by $\delta(\rho, P)$ the greatest value of the parameter δ for which the domain D_δ still lies in the domain $D_{\rho,P}$. It is obvious that

$$\begin{aligned} \inf \int_R \left| \frac{\alpha(S)}{1 + \lambda\alpha(S)} e^{i(P S)} - \Phi_\rho(S) \right|^2 [1 + \lambda\alpha(S)] d\sigma_S &\leq \\ &\leq \inf \int_R \left| \frac{\alpha(S)}{1 + \lambda\alpha(S)} - \Phi_{\delta(\rho, P)}(S) \right|^2 [1 + \lambda\alpha(S)] d\sigma_S. \end{aligned}$$

It is not hard to show that the orthogonal system constructed by us is complete, so that the last integral tends to 0 as $\delta(\rho, P) \rightarrow \infty$. But $\delta(\rho, P) \rightarrow +\infty$ when $\rho - \rho(P) \rightarrow +\infty$. From this we easily obtain the desired asymptotic formula.

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Note: Figure translations are in progress. See original paper for figures.

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