RESTRICTION OF A REPRESENTATION OF THE COMPLEMENTARY SERIES OF THE COMPLEX UNIMODULAR GROUP OF SECOND ORDER TO A REAL SUBGROUP

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.88

MATHEMATICS

B. D. ROMM

RESTRICTION OF A REPRESENTATION OF THE COMPLEMENTARY SERIES OF THE COMPLEX UNIMODULAR GROUP OF SECOND ORDER TO A REAL SUBGROUP

(Presented by Academician I. M. Vinogradov, 16 IX 1965)

I. In paper ([6]), representations of the principal series of the group G_0 of complex matrices of second order with determinant equal to 1 are considered. If these representations are restricted to the real subgroup G, then reducible representations arise, which there decompose into irreducible ones. In the present article a similar problem is solved for the complementary series.

As is known ([1]), a representation of the complementary series is determined by a parameter σ, $0 < \sigma < 2$, and is realized in a Hilbert space $\mathfrak{H}_\sigma$, which is the closure of the set of functions $f(z)$ on the complex plane satisfying the condition

$$(f, f) = \int |z_1 - z_2|^{-2+\sigma} f(z_1)\overline{f(z_2)}\, dz_1 dz_2 < \infty, \tag{1}$$

where $dz = dx\, dy$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. The operators V_a of the representation have the form

$$V_a f(z) = |\beta z + \delta|^{-2-\sigma} f(z\tilde{a}), \tag{2}$$

where

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad z\tilde{a} = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

For the group G the operators V_a define a certain reducible unitary representation d. The problem of the present article is to decompose the representation d into irreducible representations.

II. Consider the subrepresentation d^+ of the representation d in the subspace $\mathfrak{H}_\sigma^+$ of elements $f \in \mathfrak{H}_\sigma$ with support in the upper half-plane Z^+. The decomposition of d into irreducible representations reduces to the decomposition of the representations of the group G analogous to d^+. Therefore we first decompose the representation d^+.

Let $\mathfrak{H} = \mathfrak{H}_\sigma^+ \cap L_2^\mu(Z^+) \cap L_1^{\mu'}(Z^+)$, where the measures $\mu(z)$ and $\mu'(z)$ have the form $d\mu(z) = (\operatorname{Im} z)^\sigma dz$, $d\mu'(z) = (\operatorname{Im} z)^{\sigma/2-1} dz$. Put, for $z = i\tilde{a}$, $a \in G$,

$$\psi(a) = (\operatorname{Im} z)^{\sigma/2+1} f(z). \tag{3}$$

It is easy to see that for any $a_0 \in G$

$$\psi(aa_0) = (\operatorname{Im} z)^{\sigma/2+1} V_{a_0} f(z). \tag{4}$$

The totality of all matrices u_φ

$$u_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{5}$$

forms a subgroup U in the group G. If the subgroup $K \subset G$ consists of matrices

$$k = \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \quad \lambda > 0,$$

then the unique decomposition holds:

$$a = uk, \tag{6}$$

where $u \in U$, $k \in K$. For the Haar measure $d\mu(a)$ on the group G we have

$$d\mu(a) = d\varphi\, dz \,/\, 2\pi (\operatorname{Im} z)^2, \tag{7}$$

where $z = i\tilde{a} = i\tilde{k} = i\lambda^2 + \lambda\mu$.

If $u_1 \in U$, then $\tilde{u}u_1 = i$; therefore from the decomposition (6) it follows that

$$\psi(ua) = \psi(a) = \psi(k).$$

From formulas (3) and (7) it follows that

$$\int |\psi(a)|^2\, d\mu(a) = \int_{Z^+} |f(z)|^2 (\operatorname{Im} z)^\sigma\, dz,$$

$$\int |\psi(a)|\,d\mu(a) = \int_{Z^+} |f(z)|(\operatorname{Im} z)^{\sigma/2-1}\,dz. \tag{9}$$

Thus, $\psi(a) \in L = L_2(G) \cap L_1(G)$, where the spaces $L_p(G)$ are defined by the measure $d\mu(a)$, if in (3) $f(z) \in \mathfrak{H}$.

Let $z = i\tilde{a},\ z_1 = i\tilde{a}_1$, where $a, a_1 \in G$. Denote

$$\alpha_\sigma(a, a_1) = |z - z_1|^{-2+\sigma}/(\operatorname{Im} z\ \operatorname{Im} z_1)^{\sigma/2-1}. \tag{10}$$

According to (1), (3), and (7), for $f(z) \in \mathfrak{H}$

$$(f, f) = \int \alpha_\sigma(a, a_1)\psi(a)\overline{\psi(a_1)}\,d\mu(a)\,d\mu(a_1), \tag{11}$$

$$\alpha_\sigma(aa_0, a_1a_0) = \alpha_\sigma(a, a_1), \qquad a_0 \in G, \tag{12}$$

$$(f, f) = \int \alpha_\sigma(e, a_1) \int \psi(a)\overline{\psi(a_1 a)}\,d\mu(a)\,d\mu(a_1), \tag{13}$$

where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that if a sequence of compact sets $Q_i,\ i = 1, 2, \dots$, in G satisfies the conditions $Q_1 \subset Q_2 \subset \cdots$ and $\bigcup_i Q_i = G$, then

$$(f, f) = \lim_{n\to\infty} \int_{Q_n} \alpha_\sigma(e, a_1) \int \psi(a)\overline{\psi(a_1 a)}\,d\mu(a)\,d\mu(a_1), \tag{13a}$$

because an analogous equality holds for any summable function $\varphi(a)$ on G:

$$\int \varphi(a)\,d\mu(a) = \lim_{n\to\infty} \int_{Q_n} \varphi(a)\,d\mu(a). \tag{13b}$$

Conversely, if $\varphi(a)$ is a measurable function and the limit in the right-hand side of (13b) exists, then $\varphi(a)$ is summable and (13b) holds.

III. Let $a \to T_a$ be some irreducible unitary representation of the group G. Put

$$T_\psi = \int \psi(a) T_a^{-1}\, d\mu(a), \tag{14}$$

$\psi(a) \in L$, because $f(z) \in \mathfrak{H}$ according to (9); $\Phi(b) = \int \psi(a)\overline{\psi(ba)}\, d\mu(a)$. According to (8), $\Phi(bu_1) = \Phi(b)$; therefore $T_\Phi = T_{uT_\Phi}$ for any $u \in U$. It is clear that the inequality $T_\Phi \neq 0$ is possible only in the case of a representation of class 1.

Of the principal series described in the article [(2)], only the series C_q^0 belongs to this class. It can be realized in the space $L_2(-\infty, \infty)$ by the formula [(4)]

$$T_a f(x) = |\beta x + \delta|^{i\rho - 1} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right), \qquad a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{15}$$

The parameter $\rho \in (-\infty, \infty)$ determines the representation.

In $L_2(-\infty, \infty)$ there exists, unique up to a numerical factor, a normalized vector $f_0(x)$ satisfying the condition

$$T_u f_0 = f_0. \tag{16}$$

for all $u \in U$ (see (2), §5). In view of the one-dimensionality of the operator T_Φ in the case of the series C_q^0,

$$\operatorname{Sp} T_\Phi = (T_\Phi f_0, f_0)_1, \tag{17}$$

where $(f, f_1)_1$ is the scalar product in $L_2(-\infty, \infty)$, and has the form:

$$(f, f_1)_1 = \int_{-\infty}^{\infty} f(x)\overline{f_1(x)}\, dx.$$

According to the results of the article ([4])

$$\int \psi(a)\overline{\psi(a_1 a)}\, d\mu(a) = \frac{1}{32\pi^2}\int_0^\infty \rho \operatorname{th}\frac{\pi\rho}{2}\,(T_\psi^* T_\psi f_0, T_{a_1} f_0)_1\, d\rho =$$

$$= \frac{1}{32\pi^2}\int_0^\infty \rho \operatorname{th}\frac{\pi\rho}{2}\,(T_\psi f_0, T_\psi f_0)_1 (f_0, T_a f_0)_1\, d\rho, \tag{18}$$

because the space of vectors f_0 satisfying condition (16) is one-dimensional, so that from equalities (8) and (14) it follows that $T_\psi^* f_0 = T_{uT_\psi}^* f_0 = \nu f_0$, where $\nu = \text{const}$, and the asterisk denotes passage to the adjoint operator.

The integral with respect to ρ in (18) converges uniformly in a_1; therefore from equality (13a) we obtain

$$(f,f)=\lim_{n\to\infty}\frac{1}{32\pi^2}\int_0^\infty\int_{Q_n}\alpha_\sigma(e,a)(f_0,T_{af}0)_1\,d\mu(a)\,\rho\,\text{th}\,\frac{\pi\rho}{2}\,(T_\psi f_0,T_\psi f_0)_1\,d\rho. \tag{19}$$

IV. For almost all $a\in G$ there is a decomposition

$$a=u_\varphi\varepsilon u_{\varphi_1}, \tag{20}$$

where u_φ is determined by formula (5), and

$$\varepsilon=\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix}$$

with $t>0$ (([4]), appendix). a uniquely determines t,φ, and φ_1, if $a\neq e$, $\varphi,\varphi_1\in[0,\pi)$. In view of (16), $\nu(a)=\nu(\varepsilon)$, where $\nu(a)=(f_0,T_{af}0)$.

It is easy to show ([5]) that

$$\nu(\varepsilon)=\frac{1}{2}\left[P_{(-i\rho-1)/2}(\text{ch}\,2t)+P_{(i\rho-1)/2}(\text{ch}\,2t)\right], \tag{21}$$

where $P_\nu(z)$ is the spherical Legendre function of the 1st kind.

From equalities (12) and (20) it follows that $\alpha_\sigma(e,a)=|2\,\text{sh}\,t|^{-2+\sigma}$. If we put $y=\text{ch}\,2t$, then equality (19) may be written in the form

$$(f,f)=\lim_{n\to\infty}\frac{2^{\sigma/2-7}}{\pi}\int_0^\infty\int_{\widetilde{Q}_n}(y-1)^{-1+\sigma/2}\left[P_{(i\rho-1)/2}(y)+P_{(-i\rho-1)/2}(y)\right]\,dy\,\rho\times$$

$$\times\,\text{th}\,\frac{\pi\rho}{2}\int h(x,\rho)\overline{h(x,\rho)}\,dx\,d\rho, \tag{22}$$

where, according to (8) and (15), $h(x,\rho)=T_\varphi f_0$, $\widetilde{Q}_n=\{y:\ y=\text{ch}\,2t,\ a\in Q_n\}$.

If we put

$$K(z,x,\rho)=\frac{1}{\sqrt{\pi}}(\text{Im}\,z)^{(\sigma-1-i\rho)/2}\,|i(x-\text{Re}\,z)-\text{Im}\,z|^{i\rho-1},$$

then, according to (3) (K is the integral operator with kernel $K(z,x,\rho)$),

$$h(x,\rho) = \int_{Z^+} K(z,x,\rho) f(z)\,dz = Kf(z), \qquad f(z) \in \mathfrak{H}. \tag{23}$$

From the decomposition (20) it follows that $a^2+\beta^2+\gamma^2+\delta^2 = 2\operatorname{ch}2t$, so that

$$\widetilde{Q}_n = \left\{ y:\ y = \frac{a^2+\beta^2+\gamma^2+\delta^2}{2},\ a \in Q_n \right\}. \tag{24}$$

The inversion formula has the form

$$f(z) = \frac{1}{64\pi^2} (\operatorname{Im} z)^{-\sigma} \iint_{-\infty}^{\infty} \rho \operatorname{th}\frac{\pi\rho}{2}\, \overline{K(z,x,\rho)}\, h(x,\rho)\,dx\,d\rho. \tag{25}$$

In view of (4) and (14), from the relation $h = T_\psi f_0$ it follows that $h(x,\rho)$ passes into $T_{a_0} h(x,\rho)$ when $f(z)$ passes into $V_{a_0} f(z)$.

Theorem 1. Let $\mathfrak{H}_+$ denote the Hilbert space obtained by completing, in the scalar product

$$[h,h_1]_+ = \lim_{n\to\infty} \frac{2^{\sigma/2-7}}{\pi} \int_0^\infty \int_{\widetilde{Q}_n} (y-1)^{-1+\sigma/2} \left[P_{(i\rho-1)/2}(y) + P^*_{(-i\rho-1)/2}(y) \right] dy\,\rho$$

$$\times \operatorname{th}\frac{\pi\rho}{2} \int h(x,\rho)\overline{h_1(x,\rho)}\,dx\,d\rho$$

the set of all measurable functions $h(x,\rho)$ for which

$$(h,h) = \iint |h(x,\rho)|^2 \rho \operatorname{th}\frac{\pi\rho}{2}\,dx\,d\rho < \infty$$

and the limit on the right-hand side of equality (22) exists. Then $\mathfrak{H}_+$ is isometric to $\mathfrak{H}_\sigma^+$, and the isometry formula (22) is the Plancherel formula for the representation d^+. (23) and (25) are mutually inverse formulas, which hold when $f(z) \in \mathfrak{H}$.

V. Let now $f(z) \in \mathfrak{H}_\sigma$. Put $\varphi(z) = (f(z)+f(\bar z))/2$, $\varphi_1(z) = (f(z)-f(\bar z))/2$.

Theorem 2. Let $\mathfrak{H}(\mathfrak{H}_1)$ be the Hilbert space obtained by completing, in the scalar product $[h,h_1]$ ($[h,h_1]_1$),

$$[h,h_1] = \lim_{n\to\infty} \frac{2^{\sigma/2-6}}{\pi} \int_0^\infty \int_{\widetilde{Q}_n} \left[(y-1)^{-1+\sigma/2} + (y+1)^{-1+\sigma/2} \right] \left[P_{(i\rho-1)/2}(y) + \right.$$

$$+ P_{(-i\rho-1)/2}(y)\Big]\,dy\,\rho\ \mathrm{th}\,\frac{\pi\rho}{2}\int h(x,\rho)\overline{h_1(x,\rho)}\,dx\,d\rho,$$

$$[h,h_1]_1 = \lim_{n\to\infty}\frac{2^{\sigma/2-6}}{\pi}\int_0^\infty\int_{\widetilde{Q}_n}\left[(y-1)^{-1+\sigma/2}-(y+1)^{-1+\sigma/2}\right]\Big[P_{(i\rho-1)/2}(y)+$$

$$+ P_{(-i\rho-1)/2}(y)\Big]\,dy\,\rho\ \mathrm{th}\,\frac{\pi\rho}{2}\int h(x,\rho)\overline{h_1(x,\rho)}\,dx\,d\rho$$

the set $M(M_1)$ of all measurable functions $h(x,\rho)$ for which $(h,h)<\infty$, $[h,h]<\infty$, $((h,h)<\infty,\ [h,h]_1<\infty)$. Then the Plancherel formula for the representation d holds:

$$(f,f) = [h,h] + [h_1,h_1]_1,$$

where $h(x,\rho)=K\varphi(z)$, $h_1(x,\rho)=K\varphi_1(z)$, and $\varphi(z)$ and $\varphi_1(z)$ belong to certain dense sets, which are defined analogously to $\mathfrak{H}$. There is an inversion formula of the form (25).

Moscow Institute of Physics and Technology

Received
9 IX 1965

CITED LITERATURE

1. M. A. Naimark, *Linear Representations of the Lorentz Group*, Moscow, 1958.

2. V. Bargmann, Ann. Math., **47**, 568 (1948).

3. Harish-Chandra, Proc. Nat. Acad. Sci. U.S.A., **38**, No. 4, 855 (1952).

4. D. B. Romm, Izv. AN SSSR, ser. matem., **29**, No. 5 (1965).

5. N. N. Lebedev, *Special Functions and Their Applications*, Moscow, 1963.

6. B. D. Romm, DAN, **152**, No. 1 (1963).

Note: Figure translations are in progress. See original paper for figures.