ON THE SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY ALGEBRAIC POLYNOMIALS

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Abstract

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V. N. MALOZEMOV

ON THE SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY ALGEBRAIC POLYNOMIALS

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The following assertion generalizes A. F. Timan's theorem on the approximation of a function given on a finite interval by algebraic polynomials ([1]), to the case of simultaneous approximation of a function and its derivatives. It also strengthens the corresponding results of G. Freud ([2]) and A. O. Gelfond ([3]).

Theorem. *For every function $f(x)$ having a continuous r-th derivative on the finite interval $[-1, 1]$, and for every natural number $n \geq r$, one can indicate an algebraic polynomial $Q_n(x)$ of degree not exceeding n such that, for all $s = 0, 1, 2, \dots, r$; $-1 \leq x \leq 1$,*

$$\left|f^{(s)}(x) - Q_n^{(s)}(x)\right| \leq C_r \left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right]^{r-s} \omega\left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right],$$

where $\omega(\delta) = \omega(f^{(r)}; \delta)$; $\delta \in [0, \infty)$ is the modulus of continuity of $f^{(r)}(x)$; C_r is a constant depending only on r.

The polynomial $Q_n(x)$ can be written explicitly. To this end, consider the kernels $U_{N,s}(t)$ $(s = 0, 1, 2, \dots, r;\ N = 1, 2, \dots)$

$$U_{N,s}(t) = \frac{1}{\psi_N^{(s)}} \left(\frac{\sin Nt/2}{N \sin t/2}\right)^{2s+4},$$

where

$$\psi_N^{(s)} = \int_{-\pi}^{\pi} \left(\frac{\sin Nt/2}{N \sin t/2}\right)^{2s+4} dt.$$

Next put

$$P_{N,s}(f;x) = \int_{-\pi}^{\pi} f(\cos t)\, U_{N,s}(t - \arccos x)\, dt.$$

Let E be the identity operator, $N = [(n-r)/(r+2)] + 1$. Then

$$Q_n(x) = f(x) - (E - P_{N,r}) \left(\int_0^x (E - P_{N,r-1}) \times \right.$$

$$\left. \times \left(\int_0^{x_1} \cdots \left(\int_0^{x_{r-2}} (E - P_{N,1}) \left(\int_0^{x_{r-1}} (E - P_{N,0})(f^{(r)}; x_r)\, dx_r \right) dx_{r-1} \right) \cdots \right) dx_1 \right).$$

The main role in the proof of the theorem is played by the following

Lemma. *Let the function $f(x)$ have a continuous first derivative on the interval $[-1,1]$, and suppose that*

$$|f'(x)| \leq \left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right]^{\nu-1} \omega\left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right]$$

($\omega(\delta)$ is some modulus of continuity, ν is a natural number). Then for $-1 \leq x \leq 1$ the inequalities

$$\left|f(x) - P_{n,\nu}(f;x)\right| \leq M_\nu \left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right]^{\nu} \omega\left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right],$$

$$\left|P_{n,\nu}^{(s)}(f;x)\right| \leq K_\nu \left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right]^{\nu-s} \omega\left[\frac{1}{n}\sqrt{1-x^2} + \frac{1}{n}\right]$$

$$(s = 1, 2, \ldots, \nu).$$

Omitting the proof of this lemma because of its bulkiness, we pass to the proof of the theorem.

Since for all $n \geq r$

$$\frac{1}{N} = \frac{1}{n} \cdot \frac{n}{1 + [(n-r)/(r+2)]} \leq 2(r+1)\frac{1}{n},$$

it is enough to verify the relations

$$\left|f^{(s)}(x) - Q_n^{(s)}(x)\right| \leq C_r \left[\frac{1}{N}\left(\sqrt{1-x^2} + \frac{1}{N}\right)\right]^{r-s} \omega\left[\frac{1}{N}\left(\sqrt{1-x^2} + \frac{1}{N}\right)\right]$$

$$(s = 0, 1, 2, \ldots, r; \qquad -1 \leq x \leq 1).$$

First note that, by virtue of the assertion proved in (1),

$$\left|(E-P_{N,0})(f^{(r)};x)\right| \le A\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right],$$

where $\omega(\delta)=\omega(f^{(r)};\delta)$ and A is an absolute constant. The function

$$\Phi_1(x)=\int_0^x (E-P_{N,0})(f^{(r)};t)\,dt$$

has on the interval $[-1,1]$ a continuous first derivative, and moreover

$$|\Phi_1'(x)| \le A\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right].$$

By the lemma,

$$\left|(E-P_{N,1})(\Phi_1;x)\right| \le AM_1\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right]\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right],$$

$$\left|P_{N,1}'(\Phi_1;x)\right| \le AK_1\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right].$$

Hence, for $s=0,1$,

$$\left|\frac{d^s}{dx^s}(E-P_{N,1})(\Phi_1;x)\right| \le C_1\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right]^{1-s}\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right].$$

Now put

$$\Phi_2(x)=\int_0^x (E-P_{N,1})(\Phi_1;t)\,dt.$$

Analogously to the preceding, referring to the lemma, we obtain

$$\left|(E-P_{N,2})(\Phi_2;x)\right| \le C_1M_2\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right]^2\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right],$$

and for $s=1,2$

$$\left|P_{N,2}^{(s)}(\Phi_2;x)\right| \le C_1K_2\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right]^{2-s}\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right].$$

Hence, for $s=0,1,2$,

$$\left|\frac{d^s}{dx^s}(E-P_{N,2})(\Phi_2;x)\right| \le C_2\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right]^{2-s}\omega\left[\frac{1}{N}\left(\sqrt{1-x^2}+\frac{1}{N}\right)\right].$$

Continuing this process, each time setting

$$\Phi_\nu(x) = \int_0^x (E - P_{N,\nu-1})(\Phi_{\nu-1}; t)\, dt,$$

we finally obtain

$$\left| \frac{d^s}{dx^s} (E - P_{N,r})(\Phi_r; x) \right| \le C_r \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{r-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]$$

$$(s = 0, 1, 2, \dots, r).$$

It remains to note that

$$(E - P_{N,r})(\Phi_r; x) = f(x) - Q_n(x).$$

Leningrad State University
named after A. A. Zhdanov

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References Cited

[1] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960, pp. 276-280. [2] G. Szegő, *Orthogonal Polynomials*, Moscow, 1962, pp. 20-22. [3] A. O. Gelfond, UMN, 10, no. 1 (1955).

Note: Figure translations are in progress. See original paper for figures.