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The title indicates the subject matter of the two parts of the paper.

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Abstract

Full Text

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A CRITERION FOR THE EXISTENCE OF A BICOMPACT ELEMENT IN A CONTINUOUS DECOMPOSITION.

A THEOREM ON THE INVARIANCE OF WEIGHT UNDER OPEN-CLOSED FINITE-TO-ONE MAPPINGS

(Presented by Academician P. S. Aleksandrov, May 15, 1965)

The title indicates the subject matter of the two parts of the paper.

I. At the Fourth All-Union Conference of Topologists in Tashkent in 1963 I posed a number of problems connected with finding, in arbitrary continuous decompositions of spaces of broad classes, bicompact elements. At the same time I proved the following:

A. All elements of an arbitrary continuous decomposition of a complete paracompact space, except for some σ-discrete (in the topology of the decomposition space) set of them, are bicompact.

Obviously this includes the corresponding result for complete metric spaces (see ([4])).

One of the problems was formulated as follows: is assertion A true for arbitrary paracompacts? Up to now this problem has not been solved, but affirmative answers have been obtained in some important special cases. Namely:

B. Lashnev showed that an assertion analogous to A holds for arbitrary metric spaces (see ([6])).

C. For arbitrary paracompacts with the first axiom of countability the author proved the following: the cardinality of the set of non-bicompact elements of any continuous decomposition of such a paracompact does not exceed its weight ([10]).

The methods underlying the proofs of assertions A, B, C have common features, but there are also essential differences between them. The latter lead, in particular, to substantial differences in the ranges of applicability of these methods. The aim of the author in the first part of the present paper is to establish the following result, which considerably generalizes assertion B.

Theorem 1. *Let X be a regular topological space with a symmetrix d and let $f : X \to Y$ be a continuous closed mapping onto a T_1-space Y. Then $Y = Y_0 \cup Y_1$, where Y_1 is σ-discrete in Y and $f^{-1}y$ is bicompact for every point $y \in Y_0$.* **

Obviously, the following holds.

General Lemma. *Let $\{F_\alpha \mid \alpha \in M\}$ be the family of all elements of some continuous decomposition of a space X. Let, further, $\xi = \{U_\alpha \mid \alpha \in M\}$ be such a family of open subsets of X that $U_\alpha \supset F_\alpha$ for every $\alpha \in M$. Denote by M_ξ the set of all $\alpha \in M$ for each of which there exists some $\alpha' \neq \alpha$ such that $U_{\alpha'} \supset F_\alpha$. Then, whatever $\alpha \in M$ may be, the relation*

$$\langle U_\alpha \rangle \cap F_{\alpha'} \neq \Lambda$$

can hold for at most one α' from $M \setminus M_\xi$, and, consequently, the set

$$\{(F_\alpha) \mid \alpha \in M \setminus M_\xi\} = Y \setminus Y_\xi$$

*(where $Y_\xi = \{(F_\alpha) \mid \alpha \in M_\xi\}$) **** is discrete in the decomposition space (for $\bigcup_{\alpha \in M} \langle U_\alpha \rangle = X$).*

* For the definition of a symmetrix see ([1]).
** Compare the proof of Theorem 1 below with the proof of assertion B given in ([6]).
*** $\langle U_\alpha \rangle$ is the largest distinguished open set lying in U_α.
**** (F_α) denotes the point representing the element F_α in the decomposition space.

Let us apply this general assertion to our situation. Put $M = Y$, $F_y = f^{-1}y$, $y \in Y$, $\xi_n = \{\operatorname{Int} O_{1/n}(f^{-1}y) \mid y \in Y\}^*$, $n = 1, 2, \ldots, \infty$, $Y_0 = \bigcap_{n=1}^{\infty} Y_{\xi_n}$, and $Y_1 = Y \setminus Y_0$. Then, by the lemma, Y_1 is a σ-discrete subspace of the space Y. We shall show that if $y \in Y_0$, then $f^{-1}y$ is a compact subspace. Hence, by the well-known theorem of Nemytskii ([8]), it will follow that $f^{-1}y$ is compact, and the theorem will be proved. Since $y \in \bigcap_{n=1}^{\infty} Y_{\xi_n}$, there exists a sequence $\{y_n\}$, $n = 1, 2, \ldots, \infty$, such that $y_n \in Y \setminus y$, $O_{1/n} f^{-1} y_n \supset f^{-1}y$. Consider the set $P = y \cup \bigcup_{n=1}^{\infty} y_n$. Let $P' \subseteq P$ be an arbitrary infinite subset. Then $f^{-1}P' \supseteq f^{-1}y_{n_k}$ for some infinite sequence of integers n_k; hence $[f^{-1}P'] \supseteq f^{-1}y$. It follows that $y \in [P']$. Thus, in view of the arbitrariness in the choice of the set $P' \subseteq P$, it has been proved that P is compact. But P is countable; consequently, P is a compactum and, all the more, the first axiom of countability is satisfied in P. Applying Morita' s theorem ([7]), we conclude that the boundary of the set $f^{-1}y$ in the space $f^{-1}P$ is compact. But, in view of the relation $O_{1/n} f^{-1} y_n \supseteq f^{-1}y$ and the symmetry of d, $f^{-1}y$ has no interior points relative to $f^{-1}P$. Hence $f^{-1}y$ is compact. Theorem 1 is proved.

Remark. The interest of Theorem 1 lies in the fact that it covers an essentially non-paracompact case. For example, from it and from a known result of P. S. Aleksandrov and V. V. Nemytskii it follows that assertion A is also true for spaces with a refining sequence of coverings (and the so-called Moore spaces).

V. V. Proizvolov recently proved the theorem:

G. If one bicompactum can be mapped openly and finitely-to-one onto another, then the weights of the two bicompacta are equal ** ([9]). The final step of his proof was the application of my addition theorem on weight ([2]); this was preceded by the finding, in the bicompactum being mapped, of a net ***, of cardinality equal to the weight of the image. V. Proizvolov himself observed that his argument applies not only to bicompacta, but also to all classes of spaces in which the addition formula holds–for p-spaces, for subspaces of perfectly normal bicompacta (see ([3],[5])).

The aim of the present paragraph is to establish that this restriction is inessential in the case of closed mappings. Namely, the following holds.

Theorem 2. *If a Hausdorff space X is mapped open-closed and finitely-to-one onto a topological space Y, then the weight of Y is equal to that of X.*

In the proof, evidently, only the relation weight $X \leq$ weight Y is needed. First, simplifying Proizvolov' s argument, we shall discover that in X there is a net whose cardinality is equal to the weight of the space Y. Then we shall formulate and prove the basic Lemma 4.

It is well known (and very easily proved) that

Lemma 1. *Every open exactly k-to-one mapping, $k \geq 1$, is locally homeomorphic* **.

I do not know whether the following simple

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* As usual, $O_{1/n}(f^{-1}y)$ is the set of those points of the space Y which are at distance no greater than $1/n$ from $f^{-1}y$, and $\operatorname{Int} O_{1/n}(f^{-1}y)$ is the interior of the set $O_{1/n}(f^{-1}y)$.

** This is interesting and new already for mappings onto a compactum–in this case we conclude that the preimage is also a compactum.

*** For the definition of the notion of a net see ([2]).

**** A mapping $f : X \to Y$ is called locally homeomorphic if there exists a covering of the space X by open sets, each of which f maps homeomorphically onto a set open in Y.

Lemma 2. *Let $f : X \to Y$ be a locally homeomorphic exactly k-fold mapping. Then f is closed.*

Indeed, let P be any closed subset of the space X, let $Q = fP$, and let $y \notin fP$. Then $f^{-1}y \cap P = \Lambda$. Consider the points $y_1, \dots, y_k$ of which the set $f^{-1}y$ consists. Choose, at each of them, a neighborhood not intersecting P: $O_i y_i \ni y_i$, $1 \leq i \leq k$, and require that

$$O_i y_i \cap O_j y_j = \Lambda \quad \text{for } i \neq j.$$

Then

$$V = \bigcap_{i=1}^{k} fO_i y_i$$

is a neighborhood of the point y not intersecting the set Q. Indeed, from the definition of V it follows that, for any point $y' \in V$,

$$f^{-1}y' \cap \left(\bigcup_{i=1}^{k} O_i y_i \right)$$

contains at least k distinct points. And since f is exactly k-fold, we obtain

$$f^{-1}y' \subseteq \bigcup_{i=1}^{k} O_i y_i,$$

whence $f^{-1}y' \cap P = \Lambda$, or $Q = fP \not\ni y'$. In view of the arbitrariness in the choice of the points y and y', the closedness of the mapping f is proved.

Lemma 3. *Let $f : X \to Y$ be a perfect mapping onto a space Y of weight $\leq \tau$, and let $\eta = \{U_\alpha \mid \alpha \in M\}$ be an arbitrary open cover of the space X. Then from η one can choose a subcover whose cardinality does not exceed τ.*

Consider the system

$$\tilde{\eta} = \{V_\beta \mid \beta \in L\},$$

consisting of all possible finite unions of sets from η. It is clear that, to prove the lemma, it suffices to be able to choose from $\tilde{\eta}$ a cover of the space X of cardinality not greater than τ. Since the inverse image $f^{-1}y$ of any point $y \in Y$ is bicompact, there is a $\beta_y \in L$ such that $f^{-1}y \subseteq V_\beta$; hence, from the closedness of the mapping f, it follows that the "marked kernels" of the sets V_β, $\beta \in L$, i.e. the sets

$$\langle V_\beta \rangle = f^{-1}(Y \setminus f(X \setminus V_\beta)), \quad \beta \in L,$$

cover the space X in the aggregate. It is clear that it suffices to be able to choose a cover of the space X of cardinality $\leq \tau$ from the system $\{\langle V_\beta \rangle,\ \beta \in L\}$. But this is quite simple. Put $W_\beta = f\langle V_\beta \rangle$, $\beta \in L$. Then

$$\gamma = \{W_\beta,\ \beta \in L\}$$

is a cover of the space Y. Since the weight of $Y \leq \tau$, one can choose from γ some subcover

$$\gamma' = \{W_\beta,\ \beta \in L'\}$$

such that the cardinality of $L' \leq \tau$. Then the system

$$\{f^{-1}W_\beta,\ \beta \in L'\}$$

is a cover of X of cardinality $\leq \tau$. But from the definition of $\langle V_\beta \rangle$ it follows that

$$\langle V_\beta \rangle = f^{-1}W_\beta, \quad \beta \in L.$$

Thus,

$$\{\langle V_\beta \rangle,\ \beta \in L'\}$$

is the desired cover of the space X. Lemma 3 is proved.

Now let $f : X \to Y$ be an open finite-fold mapping and let the weight of $Y \leq \tau$. Denote by Y_n, $n = 1, 2, \dots, \infty$, the set of those points $y \in Y$ for which $f^{-1}y$ consists exactly of n points, and put $X_n = f^{-1}Y_n$, $f_n = f|X_n$, $f_n : X_n \to Y_n$. Then f_n is open, since f is open and X_n is the full inverse image of the space Y_n, and exactly n-fold. Moreover,

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Applying successively Lemmas 1, 2, 3 to f_n, and taking into account that the weight of Y_n does not exceed the weight of Y, we conclude that the system of open subsets of the space X_n on which f_n is homeomorphic forms a cover of the space X_n, from which one can choose some subcover

$$\lambda_n = \{G_\alpha^n \mid \alpha \in M_n\}$$

of cardinality $\leq \tau$, $n = 1, 2, \dots, \infty$. But in each G_α^n there exists a base B_α^n of cardinality $\leq \tau$ (since G_α^n is homeomorphic to a subspace of the space Y). Setting

$$B^n = \bigcup_{\alpha \in M_n} B_\alpha^n,$$

we obtain a base B^n in all of X_n of cardinality $\leq \tau$. Then

$$S = \bigcup_{n=1}^{\infty} B^n$$

is a network in X of cardinality $\leq \tau$.

Main Lemma 4 [(2)]. *If a Hausdorff space X has a network $S = \{P_\alpha,\ \alpha \in M\}$ of cardinality $\leq \tau$, and $f : X \to Y$ is a perfect mapping of the space X onto a space Y whose weight does not exceed τ, then the weight of X also does not exceed τ.*

Proof. Let

$$B = \{U_\alpha,\ \alpha \in L\}$$

be a base of Y of cardinality $\leq \tau$,

$$V_\alpha = f^{-1}U_\alpha$$

and

$$D = \{V_\alpha,\ \alpha \in L\}.$$

To each pair $P_{\alpha_1}, P_{\alpha_2} \in S$, for to which this is possible (we shall call such a pair distinguished), we associate some completely determined pair of open sets in X, $G_{\alpha_1}, G_{\alpha_2}$, for which $P_{\alpha_1} \subseteq G_{\alpha_1}$, $P_{\alpha_2} \subseteq G_{\alpha_2}$, and $G_{\alpha_1} \cap G_{\alpha_2} = \Lambda$.

Let the totality of the selected sets be denoted by E. The cardinality of the system $D \cup E^*$ does not exceed τ. Consequently, the cardinality of the "algebraic closure" of the system $D \cup E$, i.e. of the system A, consisting of the elements of $D \cup E$, complements to the closures of these elements, and all possible finite intersections of the indicated sets, is not greater than τ. Thus all elements of A are open sets.

Let us prove that A is a base of the space X. Let a point $x_0 \in X$ and its neighborhood Ox_0 be chosen arbitrarily. Put $y_0 = fx_0$ and $f^{-1}y_0 = F$. By hypothesis, F is bicompact and $F \ni x_0$. Since X is a Hausdorff space, for the point x_0 and any point $x \in \Phi = F \setminus Ox_0$ there exist disjoint neighborhoods, and hence there will also be a distinguished pair $P_{\alpha_1}, P_{\alpha_2}$ of elements of the net, $P_{\alpha_1} \ni x_0$, $P_{\alpha_2} \ni x$. Therefore, in $E \subseteq A$, for each point $x \in \Phi$ one can find such Gx_0 and Gx that $Gx_0 \ni x_0$, $Gx \ni x$, and $Gx_0 \cap Gx = \Lambda$. The family $\{Gx,\ x \in \Phi\}$ forms an open cover of Φ. Since Φ is bicompact, in this cover there is a finite subcover $\varphi : Gx_1, \dots, Gx_k$, and

$$\left[\bigcup_{i=1}^{k} Gx_i\right] = \bigcup_{i=1}^{k}[Gx_i] \not\ni x_0, \qquad Gx_i \in E.$$

Put

$$OF = Ox_0 \cup \left(\bigcup_{i=1}^{k} Gx_i\right).$$

Obviously OF is a neighborhood of F in X. From the closedness of f it follows that the point y_0 has a neighborhood Oy_0 such that $f^{-1}Oy_0 \subseteq OF$. Since B is a base of Y, there is $U_{\alpha_0} \in B$, $y_0 \in U_{\alpha_0} \subseteq Oy_0$. Then $F \subseteq V_{\alpha_0} \subseteq OF$, and $V_{\alpha_0} \in D \subseteq A$.

Put

$$O_1x_0 = V_{\alpha_0} \cap \left(Y \setminus \left[\bigcup_{i=1}^{k} Gx_i\right]\right).$$

From the relations given above it follows that $O_1x_0 \ni x_0$, and from the definition of A that $O_1x_0 \in A$.

We have

$$O_1x_0 \subseteq Ox_0 \cup \left(\bigcup_{i=1}^{k} Gx_i\right) \quad \text{and} \quad O_1x_0 \cap \left(\bigcup_{i=1}^{k} Gx_i\right) \subseteq O_1x_0 \cap \left[\bigcup_{i=1}^{k} Gx_i\right] = \Lambda.$$

Consequently, $O_1x_0 \subseteq Ox_0$. Hence, in view of the arbitrariness in the choice of x_0 and Ox_0, A is a base of X. Earlier we saw that the cardinality of A is not greater than τ. The lemma is proved. From the main lemma and the conclusion preceding its formulation there follows not only Theorem 2, but also the following more general assertion.

Theorem $2'$**.** *Let $f : X \to Y$ be an open finite-to-one mapping and $g : X \to Z$ a perfect mapping, and let both the weight of Y and the weight of Z not exceed τ. Then the weight of $X \leq \tau$.**

Unsolved problems. 1°. Let $f : X \to Y$ be an open mapping, where X is a completely regular space, Y is a space with a countable base, and $f^{-1}y$ is compact (i.e. a bicompactum with a countable base) for every point $y \in Y$. Will X then have a countable net? 2°. The same question under the additional assumption that f is closed.

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* $D \cup E$ is the totality of sets belonging to at least one of the systems D, E.

** This is a very general assertion. As a special case, it includes the addition theorem for the weight of bicompacta, for every such bicompactum is mapped perfectly onto a point.

Note: Figure translations are in progress. See original paper for figures.