ON QUASIANALYTIC CONTINUATION OF ANALYTIC FUNCTIONS THROUGH A JORDAN ARC

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Abstract

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ON QUASIANALYTIC CONTINUATION OF ANALYTIC FUNCTIONS THROUGH A JORDAN ARC

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1. Let G_1 and G_2 be nonintersecting Jordan domains (in the extended plane of the complex variable z), whose boundaries L_1 and L_2 have a common Jordan arc l. Put $G = G_1 \cup G_2$.

A set e is called *derivable* in G if, for some natural number p, $e^{(p)} \cap G = \emptyset$; here $e^{(p)}$ is the p-th derived set of the set e. We shall say that the function $f(z)$ is defined in the open set G, and write $f(z)$, $z \in G$, if $f(z)$ is defined everywhere in G except, perhaps, for a set $e = e_f$ derivable in G (depending on the function $f(z)$). The *neighborhood* of a set G is the intersection of G with an arbitrary open disk (under the assumption that this intersection is nonempty). By $r_n(z)$ we shall denote a rational function of order not exceeding n, by α_{nk}, $k = 1, \dots, m_n$, $m_n \leq n$, the poles of $r_n(z)$ (in the extended complex plane).

2. Consider the following classes of functions $f(z)$, $z \in G$, $f(z) = f_j(z)$, $z \in G_j$, $j = 1, 2$, analytic in G except, perhaps, for a set of singularities derivable in G.

Class $A(G)$. A function $f(z)$, $z \in G$, belongs to the class $A(G)$ if there exist a set e_f derivable in G and a sequence of rational functions $r_n(z)$, $n = 1, 2, \dots$, such that: a) $\{\alpha_{nk}\} \subset e_f$, $n = 1, 2, \dots$, $k = 1, 2, \dots, m_n$; b) $r_n(z)$ converges uniformly inside $G \setminus \bar{e}_f$ (i.e. on any closed set $F \subset G \setminus \bar{e}_f$, $\bar{e}_f$ being the closure of e_f) to the function $f(z)$.

It follows from Runge' s theorem that $A(G)$ can be defined as the class of all functions analytic (locally analytic) on the open set G, up to a set of singularities derivable in G. Consequently, in the class $A(G)$ the uniqueness property does not hold; every $f(z) = f_j(z)$, $z \in G_j$, $j = 1, 2$, where $f_1(z)$ and $f_2(z)$ are completely arbitrary functions analytic respectively in G_1 and G_2, belongs to the class $A(G)$.

Class $R(G)$. A function $f(z)$, $z \in G$, belongs to the class $R(G)$ if there exist a set e_f derivable in G and a sequence of rational functions $r_n(z)$, $n = 1, 2, \dots$, such that: a) $\{\alpha_{nk}\} \subset e_f$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots, m_n$; b) $r_n(z)$ converges uniformly inside $G \setminus \bar{e}_f$ to $f(z)$ with a rate characterized by the condition

$$\sup_{\{F\}} \left\{ \varlimsup_{n\to\infty} \left[\max_{z\in F} |f(z) - r_n(z)| \right]^{1/n} \right\} < 1; \tag{1}$$

the least upper bound is taken over the class of all closed sets $F \subset G \setminus \bar{e}_f$.

The simplest example of functions belonging to the class $R(G)$ is given by functions of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - \alpha_n}, \qquad z \in G,$$

where $\{\alpha_n\}' \cap G = \emptyset$ and

$$\varlimsup_{n\to\infty} \sqrt[n]{|A_n|} < 1.$$

The class $R_{\{n_i\}}(G)$, where $\{n_i\}$ is a fixed increasing sequence of natural numbers, is defined analogously to the class $R(G)$, with the difference that the upper limit in condition (1) is taken over the subsequence $\{n_i\}$.

Finally, let

$$\hat{R}(G) = \bigcup_{\{n_i\}} R_{\{n_i\}}(G).$$

Clearly, whatever the sequence $\{n_i\}$, the inclusions

$$R(G) \subset R_{\{n_i\}}(G) \subset \hat{R}(G) \subset A(G)$$

hold.

We note that the class $R(G)$, like $A(G)$, is a functional algebra; the product of functions belonging to $R(G)$ also belongs to $R(G)$. For the classes $R_{\{n_i\}}(G)$ this property, generally speaking, does not hold; $\hat{R}(G)$ is not even a linear space (see Theorem 3).

3. **Theorem 1.** *If the functions $f(z)$ and $g(z)$ belong to the class $R_{\{n_i\}}(G)$ and $f(z) = g(z)$ on some portion of the set G, then $f(z) \equiv g(z)$, $z \in G$.*

Thus each of the classes $R_{\{n_i\}}(G)$ (in particular, $R(G)$) is a quasianalytic class of functions.

Theorem 2. *Let the function $f(z) = f_j(z)$, $z \in G_j$, $j = 1, 2$, belong to the class $R_{\{n_i\}}(G)$. If there exists a function $\tilde{f}(z)$, analytic in some neighborhood U of a point $z_0 \in l$, such that $f_1(z) = \tilde{f}(z)$ for $z \in U \cap G_1$, then $f(z) = \tilde{f}(z)$ for $z \in U \cap G_2$.*

It is interesting to compare the classes of functions introduced above with the classes of functions continuous on an interval $[a, b] \subset (-\infty, +\infty)$ (bearing in mind the constructive characteristics of the latter by means of the rate of approximation by polynomials). The class $A(G)$ is the analogue of the class of all functions continuous on the interval $[a, b]$. The analogue of the class of functions analytic on $[a, b]$ is the narrowest of the classes $R_{\{n_i\}}(G)$—the class $R(G)$; specifically, functions of this class may naturally be called "analytic" on the open set G (up to a removable set of singularities in G; it is easy to single out the subclass $R'(G)$, containing only functions analytic everywhere in G). Finally, the classes $R_{\{n_i\}}(G)$ are analogues of the quasianalytic classes of S. N. Bernstein.

This analogy may be supplemented by the following theorem (an analogue of A. I. Markushevich' s theorem for Bernstein quasianalytic classes).

Theorem 3. *Every function $f(z) \in A(G)$ can be represented as the difference of two functions belonging to $\hat{R}(G)$.*

It follows from this that in $\hat{R}(G)$ the uniqueness property does not hold; however, one may assert that if $f(z) \in \hat{R}(G)$ and $f(z) = 0$ on some portion of the set G, then $f(z) \equiv 0$, $z \in G$ (or, more generally, if $f(z) \in \hat{R}(G)$, $g(z) \in R(G)$, and $f(z) = g(z)$ on a portion of the set G, then $f(z) \equiv g(z)$, $z \in G$).

Let us emphasize an essential distinction between the class $R(G)$ and the classical quasianalytic classes of functions. The Bernstein and Denjoy-Carleman classes are constructed as a result of an *extension* of the class of functions analytic on an interval, while preserving the principal property of this class—the uniqueness property. Conversely, the class $R(G)$ is constructed by *narrowing* the class of analytic functions on a disconnected set G, where the uniqueness property does not hold; the class $R(G)$ serves as a natural substitute for the class of analytic functions for such sets G. In turn, the classes $R_{\{n_i\}}(G)$ are obtained by the usual device as a result of extending $R(G)$ while preserving the uniqueness property.

4. Theorems 1 and 2 make it possible to introduce the concept of quasianalytic continuation, generalizing the concept of analytic continuation.

Let, as above, G_1 and G_2 be Jordan domains whose boundaries have a common arc l; $G = G_1 \cup G_2$. If the point $z_0 \in l$, U is a neighborhood

of the point z_0, then by $G_j^U \subset G_j$, $j = 1, 2$, we shall denote the connected component of the open set $G_j \cap U$ whose boundary contains the point z_0; set $G^U = G_1^U \cup G_2^U$.

Let the function $f(z)$ belong to $A(G)$; if there are a point $z_0 \in l$ and a sequence $\{n_i\}$ such that, for some neighborhood U of the point z_0, the function $f(z)$, $z \in G^U$, belongs to $R_{\{n\}}(G^U)$, then the function $f_2(z) = f(z)$, $z \in G_2$, will be called a **quasianalytic continuation** of the function $f_1(z) = f(z)$, $z \in G_1$ (relative to $z_0, \{n_i\}$).

For fixed z_0 and $\{n_i\}$, the quasianalytic continuation $f_2(z)$, $z \in G_2$, of the function $f_1(z)$, $z \in G_1$, is determined uniquely (if it exists); this follows from Theorem 1 and the uniqueness theorem for analytic functions. It follows from Theorem 2 that quasianalytic continuation is a generalization of analytic continuation; if $f_1(z)$, $z \in G_1$, admits an analytic continuation $f_2(z)$, $z \in G_2$, through an arc l_1, $z_0 \in l_1 \subset l$, then $f_2(z)$ is a quasianalytic continuation relative to z_0 and $n_i = i = 1, 2, ...$ (and hence to any $\{n_i\}$).

The same function $f_1(z)$, $z \in G_1$, may have different quasianalytic continuations depending on the choice of the point z_0 and of the sequence $\{n_i\}$. For fixed $n_i = i = 1, 2, ...$, the nature of the multivaluedness under quasianalytic continuation (the dependence on z_0) is entirely analogous to the possible multivaluedness under analytic continuation. Examples of functions admitting single-valued quasianalytic continuation are functions $f_1(z)$, $z \in G_1$, such that there exists $f(z) \in R(G)$, $f(z) = f_1(z)$, $z \in G_1$.

5. Applying Theorems 1 and 2 to the domains G^+ and G^-, which are respectively the interior and the exterior of a closed Jordan curve L, we obtain one of the possible solutions of the problem of quasianalytic continuation and quasianalytic classes of functions defined in two Jordan domains with a common boundary. Various variants of this problem arose in the works of Weierstrass and Borel.

Separate examples of series of rational functions defining "different" functions in different components of their domain of convergence led Weierstrass to a negative answer to the question he himself had posed about the possibility of generalizing the notion of analytic continuation through a Jordan arc. The following example is typical:

$$\frac{z}{z-1} + \frac{z}{1-z^2} + \dots + \frac{z^{2^n-1}}{1-z^{2^n}} + \dots = \begin{cases} 0, & z \in G_0^+ = \{z: \ |z| < 1\}, \\ 1, & z \in G_0^- = \{z: \ |z| > 1\}. \end{cases}$$

(Runge' s theorem was proved several years later.)

Theorem 1 shows that even a slight modification of the preceding example changes the matter qualitatively. Let $0 < q < 1$,

$$\frac{z}{z-1} + \frac{qz}{1-z^2} + \dots + \frac{(qz)^{2^n-1}}{1-z^{2^n}} + \dots = S(z) = \begin{cases} S^+(z), & z \in G_0^+, \\ S^-(z), & z \in G_0^-. \end{cases}$$

Then $S(z) \in R(G_0)$, $G_0 = \{z: \ |z| \neq 1\}$, and therefore $S(z)$ is a "single" function throughout its domain of definition; at the same time the domains G_0^+ and G_0^- are the natural domains of existence of the analytic functions $S^+(z)$ and $S^-(z)$, respectively.

Let us dwell on one of Borel' s problems. Let L be a closed Jordan curve, G^+ and G^- its interior and exterior, respectively; $G = G^+ \cup G^-$. Consider the series

$$\sum_{n=1}^{\infty} \frac{A_n}{z-\alpha_n}, \qquad \sum_{n=1}^{\infty} |A_n| < \infty, \qquad \alpha_i \neq \alpha_j,\ i \neq j, \tag{2}$$

under the conditions

$$\{\alpha_n\} \subset G^-, \qquad \{\alpha_n\}' = L; \tag{3}$$

We shall denote the class of functions representable by series of the form (2)–(3) by $B(G)$. Borel [1] posed the question of whether there exists an intrinsic connection between the components $f^+(z)$, $z \in G^+$, and $f^-(z)$, $z \in G^-$, of a function $f(z) \in B(G)$. In particular, can $f^+(z)$ admit an analytic continuation into the domain of existence of the poles of the series representing it?

In answering these questions, Wolff [2] showed that any function $f^+(z)$, analytic in the closed domain $\overline{G^+}$, can be represented in G^+ by a series of the form (2)–(3); to a fixed interior component $f^+(z)$ of a function $f(z) \in B(G)$ there correspond various (indeed, infinitely many) series of the form (2)–(3), and consequently exterior functions $f^-(z)$. Danjoy [3] strengthened these results, showing that Wolff' s assertions remain valid in the class of series (2)–(3) satisfying the condition $|A_n| < e^{-n^{1/2-\delta}}$, $\delta > 0$.

Let $B_1(G)$ denote the class of functions representable by series (2)–(3) with the condition

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} < 1.$$

It is easy to see that $B_1(G) \subset R(G)$. Consequently, prescribing a function $f(z) \in B_1$ on any portion of the set G determines $f(z)$ uniquely everywhere on this set. If, moreover, all $A_n \neq 0$, then the curve L is a singular line for $f^+(z)$, represented by the corresponding series (see Theorem 2; the fact that L is a singular line for $f^-(z)$ is obvious). We note that Borel, on the basis of the theory of monogenic functions constructed by him, obtained analogous results only under the condition

$$|A_n| < \exp(-\exp n^2).$$

6. Analogues of the results presented above are also valid for arbitrary non-intersecting domains G_1 and G_2. In the general case the quasi-analytic classes $R(G)$ and $R_{\{n_i\}}(G)$, $G = G_1 \cup G_2$, are defined analogously, with the difference that on the right-hand side of condition (1) there stands a certain number $q = q(G) \leqslant 1$; if $\overline{G_1} \cap \overline{G_2} = \emptyset$, then $q(G) < 1$. For domains G_1 and G_2 whose boundaries L_1 and L_2 have at least one point of tangency z_0, $q(G) = 1$, and the results are valid without change (cf. (4)).

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Note: Figure translations are in progress. See original paper for figures.