Soviet-era science, translated into English

THE MILNE PROBLEM WITH AN ARBITRARY PHASE FUNCTION AND TAKING ACCOUNT OF THE AZIMUTHAL INHOMOGENEITY OF THE SOLUTION

MATHEMATICS

1966

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196601.52751

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 517.947.3

MATHEMATICS

M. V. MASLENNIKOV

THE MILNE PROBLEM WITH AN ARBITRARY PHASE FUNCTION AND TAKING ACCOUNT OF THE AZIMUTHAL INHOMOGENEITY OF THE SOLUTION

(Presented by Academician M. V. Keldysh on 16 IX 1965)

The problem of the passage of radiation through a semi-infinite plane layer of matter (with inward normal to the boundary **n**) leads to the equation

$$x(\tau, \vec{\omega}) = \hat{A}x(\tau, \vec{\omega}) + \mathcal{F}(\tau, \vec{\omega}), \qquad 0 < \tau < \infty, \quad \vec{\omega} \in \Omega. \tag{1}$$

Here $x(\tau, \vec{\omega})$ is the phase density of the radiation at optical depth $\tau \in (0, \infty)$, propagating in the direction $\vec{\omega} \in \Omega$ (Ω is the unit sphere of three-dimensional space). The distribution of radiation sources and the boundary condition at $\tau = 0$ determine the form of $\mathcal{F}(\tau, \vec{\omega})$. The definition of the operator $\hat{A}$ will be given below.

We shall assume that the scattering phase function $g(\mu)$ satisfies conditions 1) of ([1]). In addition, we adopt the notation of the paper ([1]).

Let ε be a real number. Denote by B_ε the class of functions $x(\tau)$, defined on $(0, \infty)$ with values in $L_2(\Omega)$, such that: a) for $0 < \tau_1 < \infty$

$$\sup_{\tau \in (0, \tau_1)} \|x(\tau)\| < \infty;$$

b) $e^{-\varepsilon\tau}x(\tau)$ is summable on $(0, \infty)$ in the Bochner sense ([2]). To distinguish a function from B_ε from its value $x(\tau) \in L_2(\Omega)$ at the point τ, we shall write $x(\cdot)$; $x(\tau, \vec{\omega})$ is the value assumed by the function $x(\tau) \in L_2(\Omega)$ at the point $\vec{\omega} \in \Omega$. If $x(\cdot) \in B_\varepsilon$, then $y(\tau) = \hat{g}x(\tau) \in C(\Omega)$ ([1]), $\tau \in (0, \infty)$, and, as a function of τ with values in $C(\Omega)$, $y(\tau)$ is strongly measurable ([2]) on $(0, \infty)$. In particular, for fixed $\vec{\omega}$, $\hat{g}x(\tau, \vec{\omega}) \equiv y(\tau, \vec{\omega})$ is a numerical Lebesgue-measurable function of $\tau \in (0, \infty)$.

Definition. For $x(\cdot) \in B_1$, $\hat{A}x(\cdot) = z(\cdot)$, where, for $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, $z(\tau, \vec{\omega})$ is determined by the relations:

SovietRxiv

$$z(\tau,\omega)=\begin{cases}\dfrac{1}{\vec{\omega}\mathbf{n}}\displaystyle\int_0^{\tau}\exp\Big(-\frac{\tau-\rho}{\vec{\omega}\mathbf{n}}\Big)\,\hat{g}x(\rho,\vec{\omega})\,d\rho, & \text{if } \vec{\omega}\mathbf{n}>0;\\ \hat{g}x(\tau,\vec{\omega}), & \text{if } \vec{\omega}\mathbf{n}=0;\\ \dfrac{1}{|\vec{\omega}\mathbf{n}|}\displaystyle\int_{\tau}^{\infty}\exp\Big(-\frac{\rho-\tau}{|\vec{\omega}\mathbf{n}|}\Big)\,\hat{g}x(\rho,\vec{\omega})\,d\rho, & \text{if } \vec{\omega}\mathbf{n}<0.\end{cases}$$

It turns out that, for $x(\cdot)\in B_1$, $z(\cdot)=\hat{A}x(\cdot)$ is an abstract function of $\tau\in(0,\infty)$ with values in $L_2(\Omega)$, depending continuously on τ. For $0<\tau_1<\infty$

$$\sup_{(\tau,\vec{\omega})\in(0,\tau_1]\times\Omega}|z(\tau,\vec{\omega})|<\infty.$$

For fixed $\tau\in(0,\infty)$, $z(\tau,\vec{\omega})$ is continuous in the second argument at all points $\vec{\omega}\in\Omega$ for which $\vec{\omega}\mathbf{n}\neq 0$. If $\delta>0$, $\tau_1\in(0,\infty)$, then uniformly with respect to all those $\vec{\omega}\in\Omega$ for which $|\vec{\omega}\mathbf{n}|\geq\delta$,

$$\lim z(\tau,\vec{\omega})=$$

$$=z(\tau_1,\vec{\omega}).$$

Put

$$z_0(\vec{\omega})=\frac{1}{|\vec{\omega}\vec{n}|}\int_0^{\infty}\exp\left(-\frac{\rho}{|\vec{\omega}\vec{n}|}\right)\hat{g}x(0,\vec{\omega})\,d\rho$$

for $\vec{\omega}\vec{n}<0$, and $z_0(\vec{\omega})=0$ for $\vec{\omega}\vec{n}\geqslant 0$. Then, uniformly with respect to all $\vec{\omega}\in\Omega$, $|\vec{\omega}\vec{n}|\geqslant\delta$,

$$\lim_{\tau=0+0}z(\tau,\vec{\omega})=z_0(\vec{\omega});$$

$$\sup_{\vec{\omega}\in\Omega}|z_0(\vec{\omega})|<\infty;$$

$z_0(\vec{\omega})$ is continuous at every point $\vec{\omega}\in\Omega$ for which $\vec{\omega}\vec{n}\neq 0$; $z_0\in L_2(\Omega)$;

$$\lim_{\tau=0+0}\|z(\tau)-z_0\|=0.$$

If $\tau_0\in(0,\infty)$ is a point of strong continuity of $x(\cdot)$, then $z(\tau_0)\in C(\Omega)$, and uniformly with respect to all $\vec{\omega}\in\Omega$

$$\lim_{\tau=\tau_0}z(\tau,\vec{\omega})=z(\tau_0,\vec{\omega}).$$

If there exists the limit

$$\lim_{\tau=0+0}x(\tau)=x_0\in L_2(\Omega),$$

SovietRxiv

then let $z_{00}(\vec{\omega}) = z_0(\vec{\omega})$ for $\vec{\omega}\vec{n} \neq 0$, and $= \hat{g}x_0(\vec{\omega})$ for $\vec{\omega}\vec{n} = 0$. Then $z_{00}(\vec{\omega})$ is continuous on the closed hemisphere

$$\Omega^- = \{\vec{\omega} \mid \vec{\omega} \in \Omega,\ \vec{\omega}\vec{n} \leqslant 0\},$$

and, uniformly with respect to $\vec{\omega} \in \Omega^-$,

$$\lim_{\tau=0+0} z(\tau, \vec{\omega}) = z_{00}(\vec{\omega}).$$

Theorem 1. For $\varepsilon \in (-1, 1)$, $\hat{A}(B_\varepsilon) \subset B_\varepsilon$; $\hat{A}(B_1) \subset B_1$. If $x(\cdot) \in B_\varepsilon$, $\varepsilon \in (-1, 1)$, then there exist constants $k_{3,4}$, $0 < k_{3,4} < \infty$, such that

$$\|\hat{A}^3 x(\tau)\| \leqslant k_3 e^{\varepsilon\tau}, \qquad |\hat{A}^4 x(\tau, \vec{\omega})| \leqslant k_4 e^{\varepsilon\tau}$$

for all $\tau \in (0, \infty)$ and $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, respectively.

Let $Y(\xi) = 0$ for $\xi \leqslant 0$, $Y(\xi) = 1$ for $\xi > 0$;

$$1(\tau, \vec{\omega}) \equiv 1,$$

$$p_0(\tau, \vec{\omega}) = Y(\vec{\omega}\vec{n}) \exp(-\tau/\vec{\omega}\vec{n})$$

for $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, and let F be an arbitrary bounded closed subset of the set $(0, \infty)$.

Theorem 2. Uniformly with respect to all $(\tau, \vec{\omega}) \in F \times \Omega$,

$$\lim_{n=\infty} g_0^{-n} \hat{A}^n 1(\tau, \vec{\omega}) = 0, \qquad \sum_{\nu=0}^{\infty} g_0^{-\nu} \hat{A}^\nu p_0(\tau, \vec{\omega}) = 1.$$

Theorem 3. Let $\psi \in L_2(\Omega)$; $\psi \neq \vartheta$; let λ and s be complex numbers, $\lambda_1 \equiv \operatorname{Re}\lambda < 1$, $\lambda \in Z_0$, $|s| \geqslant g_0$, and

$$s(1 + \lambda\vec{\omega}\vec{n})\psi(\vec{\omega}) = \hat{g}\psi(\vec{\omega}), \qquad \vec{\omega} \in \Omega.$$

This means that $\psi \in C(\Omega)$ and

$$\psi^+ = \sup_{\vec{\omega}\in\Omega} |\psi(\vec{\omega})| < \infty.$$

Put

$$x(\tau, \vec{\omega}) = e^{\lambda\tau}\psi(\vec{\omega}), \qquad p(\tau, \vec{\omega}) = \psi(\vec{\omega}) p_0(\tau, \vec{\omega}).$$

Then $x(\cdot) \in B_\varepsilon$ for $\varepsilon > \lambda_1$, $p(\cdot) \in B_\varepsilon$ for $\varepsilon > -1$.

I. Uniformly with respect to $(\tau, \vec{\omega}) \in F \times \Omega$, there exists the limit

$$\bar{x}(\tau, \vec{\omega}) = \lim_{n=\infty} s^{-n} \hat{A}^n x(\tau, \vec{\omega}).$$

SovietRxiv

Moreover

$$\bar{x}(\tau,\vec{\omega}) = x(\tau,\vec{\omega}) - \sum_{\nu=0}^{\infty} s^{-\nu}\hat{A}^{\nu}p(\tau,\vec{\omega}).$$

The series on the right converges absolutely and uniformly with respect to $(\tau,\vec{\omega}) \in F \times \Omega$;

$$|\bar{x}(\tau,\vec{\omega}) - x(\tau,\vec{\omega})| \leqslant \psi^{+}$$

for all $(\tau,\vec{\omega}) \in (0,\infty)\times\Omega$.

II. $\bar{x}(\tau,\vec{\omega})$ is a continuous function of $(\tau,\vec{\omega}) \in (0,\infty)\times\Omega$; $\bar{x}(\cdot) \in B_{\varepsilon}$ for $\varepsilon > \lambda_1$.

III. If $\lambda_1 > 0$, then there exists $\tau_1 > 0$ such that $\bar{x}(\tau) \neq \vartheta$ for $\tau > \tau_1$. If $\lambda_1 \leqslant 0$, then $\bar{x}(\tau) \equiv \vartheta$ for $\tau \in (0,\infty)$.

IV. $\hat{A}\bar{x}(\tau) = s\bar{x}(\tau)$, $\tau \in (0,\infty)$.

Theorem 4. Let h_1 and h_2 be constants and, for $(\tau,\vec{\omega}) \in (0,\infty)\times\Omega$,

$$y(\tau,\vec{\omega}) = \tau + h_1 + h_2\vec{\omega}\vec{n}.$$

Then $y(\cdot) \in B_{\varepsilon}$ for $\varepsilon > 0$.

I. Uniformly with respect to $(\tau,\vec{\omega}) \in F\times\Omega$, there exists the limit

$$\bar{y}(\tau,\vec{\omega}) = \lim_{n=\infty} g_0^{-n}\hat{A}^{n}y(\tau,\vec{\omega});$$

$\bar{y}(\tau,\vec{\omega})$ does not depend on the choice of $h_{1,2}$ and

$$\bar{y}(\tau,\vec{\omega}) = \tau + g_0\frac{1-\vec{\omega}\mathbf{n}}{g_0-g_1} - \sum_{\nu=0}^{\infty} g_0^{-\nu}\hat{A}^{\nu}q(\tau,\vec{\omega}), \qquad 2\partial e\, q(\tau,\vec{\omega}) = g_0(g_0-g_1)^{-1}\times$$

$$\times (1-\vec{\omega}\mathbf{n})p_0(\tau,\vec{\omega}).$$

The series on the right converges uniformly with respect to $(\tau,\vec{\omega}) \in F\times\Omega$, and for all $(\tau,\vec{\omega}) \in (0,\infty)\times\Omega$

$$0 \leqslant \sum_{\nu=0}^{\infty} g_0^{-\nu}\hat{A}^{\nu}q(\tau,\vec{\omega}) \leqslant g_0(g_0-g_1)^{-1}.$$

II. $\bar{y}(\tau,\vec{\omega})$ is continuous in $(\tau,\vec{\omega})$ and nonnegative on $(0,\infty)\times\Omega$, $\bar{y}(\cdot) \in B_{\varepsilon}$ for $\varepsilon > 0$.

III. $\hat{A}\bar{y}(\tau) = g_0\bar{y}(\tau)$, $\tau > 0$.

Theorem 5. Let $g_0 < 1$, $\varepsilon \in (-\lambda_0,\lambda_0)$ (respectively $g_0 \leqslant 1$, $\varepsilon \in (-1,0]$), $x(\cdot) \in B_{\varepsilon}$. Then:

I. The series

$$s(\tau,\vec{\omega}) = \sum_{n=0}^{\infty} \hat{A}^{n}x(\tau,\vec{\omega})$$

converges uniformly on $F\times\Omega$.

SovietRxiv

II. $s(\tau) = x_1(\tau) + u(\tau)$, $\tau > 0$, where $x_1(\cdot) \in B_\varepsilon$, $u(\tau, \vec{\omega})$ is continuous in $(\tau, \vec{\omega})$ on $(0, \infty) \times \Omega$;

$$\sup\{e^{-\varepsilon\tau}|u(\tau, \vec{\omega})| \mid (\tau, \vec{\omega}) \in (0, \infty) \times \Omega\} < \infty$$

(respectively, for every $\varepsilon' \in (\varepsilon, 0] \cap [-\lambda_0, 0]$,

$$\sup\{e^{-\varepsilon'\tau}|u(\tau, \vec{\omega})| \mid (\tau, \vec{\omega}) \in (0, \infty) \times \Omega\} < \infty).$$

III. $s(\cdot) \in B_{\varepsilon'}$ for $\varepsilon' > \varepsilon$ (respectively, for $\varepsilon' > \max\{\varepsilon, -\lambda_0\}$).

IV. $s(\tau) = \hat{A}s(\tau) + x(\tau)$, $\tau > 0$.

Let $k \in \mathfrak{R}$, $k > 0$ (1). Put in Theorem 3 $\lambda = k$, $s = 1$, $\psi = \psi_{kp}$, $x_{kp}(\tau) = e^{k\tau}\psi_{kp}$. Theorem 3 assigns to the function $x(\cdot) = x_{kp}(\cdot)$ the function $\bar{x}_{kp}(\cdot) = \bar{x}(\cdot)$. In the same way we define, for $g_0 = 1$,

$$\bar{y}_0(\cdot) = \bar{y}(\cdot),$$

where $\bar{y}(\cdot)$ is the function from Theorem 4. Let

$$X = \{\bar{x}_{kp}(\cdot) \mid k \in \mathfrak{R},\ k > 0,\ p = 1, 2, \ldots, p_k\}$$

and, for $g_0 = 1$,

$$X_1 = X \cup \{\bar{y}_0(\cdot)\}.$$

Then $X \subset \bar{B} = \bigcup_{\varepsilon<1} B_\varepsilon$ and, for $g_0 = 1$, $X_1 \subset \bar{B}$. It is obvious that X (and for $g_0 = 1$, X_1) is a linearly independent system of functions. The only sign-constant function from X (for $g_0 = 1$, from X_1) is $\bar{x}_{\lambda_0 1}(\cdot)$ (respectively $\bar{y}_0(\cdot)$).

We shall call **problem A** problem (1) with

$$\mathcal{F}(\tau, \vec{\omega}) = B(\vec{\omega}) \exp(-\tau/\vec{\omega}\mathbf{n}),$$

where $B \in L_2(\Omega)$, $B(\vec{\omega}) = 0$ for $\vec{\omega}\mathbf{n} \leqslant 0$. Problem A is the problem of the diffusion of radiation through a semi-infinite layer, on whose boundary an externally incident flux is distributed according to the law $B(\vec{\omega})$. If $B = 0$, then problem A becomes homogeneous.

Theorem 6. In the class of functions $x(\cdot) \in \bar{B}$, each of which satisfies one of the conditions: a) $x(\cdot) \in B_\varepsilon$ for some $\varepsilon \in (-1, \lambda_0) \cup (-1, 0]$ or b)

$$\sup_{\tau>0} \|x(\tau)\| < \infty,$$

there exists exactly one solution of problem A. This solution $\bar{s}(\tau)$ is represented by the Neumann series

$$\bar{s}(\tau, \vec{\omega}) = \sum_{n=0}^{\infty} \hat{A}^n \mathcal{F}(\tau, \vec{\omega}),$$

SovietRxiv

converging uniformly on $F \times \Omega$.

$$\sup_{\tau>0} \|\bar{s}(\tau)\| < \infty; \qquad \bar{s}(\cdot) \in B_{\varepsilon'}$$

for $\varepsilon' \in (-\lambda_0, 1)$.

Let $x(\cdot) \in B_\varepsilon$. Then for $\operatorname{Re} k > \varepsilon$ the function

$$\tilde{x}(k) = \int_0^\infty e^{-k\tau} x(\tau)\, d\tau$$

is defined and holomorphic, with values in $L_2(\Omega)$. The inversion formulas for this Laplace-Bochner integral depend on additional properties of $x(\cdot)$. Such properties are possessed by $x(\cdot)$ if it is a solution of problem A.

Theorem 7. Let $\varepsilon \in (-1, 1)$, $x(\cdot) \in B_\varepsilon$, and let $x(\cdot)$ be a solution of problem A. Then

$$\sup_{\tau>0} e^{-\varepsilon\tau} \|x(\tau)\| < \infty;$$

there exists the limit

$$\lim_{\tau=0+0} x(\tau) =$$

$= x_0 \in L_2(\Omega)$, $\quad x_0(\vec{\omega}) = B(\vec{\omega})$ for $\vec{\omega}\vec{n} > 0$; the Laplace transform $\tilde{x}(k)$, for $\operatorname{Re} k > \varepsilon$, satisfies the equation

$$(1 + k\vec{\omega}\vec{n})\tilde{x}(k) = g\tilde{x}(k) + (\vec{\omega}\vec{n})x_0.$$

Relying on (1), we can now analytically continue $\tilde{x}(k)$ to Z_0. The singularities of $\tilde{x}(k)$ are the points $\mathfrak{R} \subset (-1, 1)$. It can be shown that for large $|\operatorname{Im} k|$,

$$\|\tilde{x}(k)\| \leq \text{const} \cdot |\operatorname{Im} k|^{-1}.$$

This last circumstance makes it possible, using a suitable inversion formula for the Laplace integral, to obtain the following representations of $x(\tau)$. Let

$$s \in (-1, \min\{0, \varepsilon\}) \setminus \mathfrak{R}.$$

Then: a) for $g_0 < 1$ and for $g_0 = 1$, $\varepsilon < 0$,

$$x(\tau) = -\sum_{\lambda \in \mathfrak{R} \cap (s,\varepsilon]} e^{\lambda\tau} \operatorname{sgn} \lambda \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p})\psi_{\lambda p} + e^{s\tau} r_s(\tau) =$$

$$= -\sum_{\lambda \in \mathfrak{R} \cap (0,\varepsilon]} \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p})\bar{x}_{\lambda p}(\tau) + \bar{s}(\tau);$$

SovietRxiv

b) for $g_0 = 1,\ \varepsilon \geq 0$,

$$x(\tau) = -\frac{3}{4\pi}(1-g_1)(x_0,\vec{\omega}\vec{n})\tau + \frac{3}{4\pi}\left[(x_0,\vec{\omega}\vec{n})\vec{\omega}\vec{n} + (x_0,(\vec{\omega}\vec{n})^2)\right] -$$

$$-\sum_{\substack{\lambda\in\mathfrak{R}\cap(s,\varepsilon]\\ \lambda\neq 0}} e^{\lambda\tau}\operatorname{sgn}\lambda \sum_{p=1}^{p_\lambda}(x_0,(\vec{\omega}\vec{n})\psi_{\lambda p})\psi_{\lambda p} + e^{s\tau}r_s(\tau) =$$

$$= -\sum_{\lambda\in\mathfrak{R}\cap(0,\varepsilon]}\sum_{p=1}^{p_\lambda}(x_0,(\vec{\omega}\vec{n})\psi_{\lambda p})\bar{x}_{\lambda p}(\tau) - \frac{3}{4\pi}(1-g_1)(x_0,\vec{\omega}\vec{n})\bar{y}_0(\tau) + \bar{s}(\tau).$$

In both cases

$$\sup_{\tau>0}\|r_s(\tau)\| < \infty.$$

Theorem 8. Let $x(\cdot)\in\bar{B}$. $x(\cdot)$ is a solution of the homogeneous problem A if and only if $x(\cdot)$ is a linear combination of a finite number of functions from the system X (for $g_0 < 1$) or X_1 (for $g_0 = 1$). $x(\cdot)$ is a solution of the inhomogeneous problem A if and only if

$$x(\tau) = x_1(\tau) + \bar{s}(\tau),$$

where $x_1(\cdot)\in\bar{B}$, and $x_1(\cdot)$ is a solution of the homogeneous problem A.

In order that problem A admit nonnegative solutions, it is necessary and sufficient that

$$B(\vec{\omega}) \geq 0 \quad \text{for } \vec{\omega}\in\Omega.$$

If this condition is fulfilled, then Theorem 9 is true.

Theorem 9. Let $x_1(\cdot)\in\bar{B}$, and let $x_1(\cdot)$ be a nonnegative solution of problem A. Then

$$x_1(\tau) = x(\tau) + \bar{s}(\tau),$$

where $x(\cdot)\in\bar{B}$, and $x(\cdot)$ is a nonnegative solution of the homogeneous problem A. Moreover: a) if $g_0 < 1$, then (cf. (1))

$$x(\tau) = C\bar{x}_{\lambda_0 1}(\tau) = \frac{(x_0,(\vec{\omega}\vec{n})\Phi_{\lambda_0})}{(\vec{\omega}\vec{n},\Phi^2_{\lambda_0})}e^{\lambda_0\tau}\Phi_{\lambda_0} - \frac{(x_0,(\vec{\omega}\vec{n})\Phi_{-\lambda_0})}{(\vec{\omega}\vec{n},\Phi^2_{\lambda_0})}e^{-\lambda_0\tau}\Phi_{-\lambda_0} + e^{-\lambda_1\tau}\rho(\tau);$$

b) if $g_0 = 1$, then

$$x(\tau) = C\bar{y}_0(\tau) = -\frac{3}{4\pi}(1-g_1)(x_0,\vec{\omega}\vec{n})\tau +$$

$$+\frac{3}{4\pi}\left[(x_0,\vec{\omega}\vec{n})\vec{\omega}\vec{n} + (x_0,(\vec{\omega}\vec{n})^2)\right] + e^{-\lambda_1\tau}\rho(\tau).$$

SovietRxiv

In both cases

$$0 \le C = \mathrm{const} < \infty, \qquad \sup_{\tau > 0} \|\rho(\tau)\| < \infty,$$

$$\lambda_1 = \min(\mathfrak{R}_0 \setminus \{\lambda_0\}) \quad (\text{cf. } (1)),$$

if $\mathfrak{R}_0$ contains more than one point, and λ_1 is an arbitrary number from $(\lambda_0, 1)$ otherwise.

Received
14 IX 1965

References Cited

1. M. V. Maslennikov, *DAN*, **168**, No. 3 (1966).

2. E. Hille, R. Phillips, *Functional Analysis and Semigroups*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.